

TIGHT BINDING BOOK

UNIVERSAL  
LIBRARY

**OU\_164432**

UNIVERSAL  
LIBRARY



OSMANIA UNIVERSITY LIBRARY

Call No. 513.21 <sup>1M 11T</sup> Accession No. 13447

Author M. Clelland - 5

Title Treatise on the Geometry of the

This book should be returned on or before the date  
last marked below. Circle... Reciprocalism

---





A TREATISE  
ON THE  
GEOMETRY OF THE CIRCLE.



A TREATISE  
ON THE  
GEOMETRY OF THE CIRCLE  
AND SOME EXTENSIONS TO  
CONIC SECTIONS BY THE METHOD OF  
RECIPROCATION /

*WITH NUMEROUS EXAMPLES*

BY  
WILLIAM J. M'CLELLAND, M.A.  
PRINCIPAL OF THE INCORPORATED SOCIETY'S SCHOOL, SANTRY, DUBLIN

London  
MACMILLAN AND CO.  
AND NEW YORK  
1891

*All rights reserved*



## PREFACE.

MY object in the publication of a treatise on Modern Geometry is to present to the more advanced students in public schools and to candidates for mathematical honours in the Universities a concise statement of those propositions which I consider to be of fundamental importance, and to supply numerous examples illustrative of them.

Results immediately suggested by the propositions, whether as particular cases or generalized statements, are appended to them as Corollaries.

The Examples are printed in smaller type, and are classified under the Articles containing the principal theorems required in their solution.

The more difficult ones are fully worked out, and in most cases hints are given to the others.

The reader who is familiar with the first six books of Euclid with easy deductions and the elementary formulæ in Plane Trigonometry will thus experience little difficulty in mastering the following pages.

I have dwelt at length in Chap. II. on the Theory of Maximum and Minimum.

Chap. III. is devoted to the more recent developments of the geometry of the triangle, initiated in 1873 by Lemoine's paper entitled "*Sur quelques propriétés d'un point remarquable du triangle.*"

The study of the Brocardian Geometry is appropriate at this stage, as I have shown that the deductions of M. Brocard and of other geometers, both in England and on the Continent, are simple and direct inferences of the well-known property of Art. 19, which has been called the Point *O* Theorem.

Chap. IX. gives an account of the researches of Neuberg and Tarry on Three Similar Figures.

A feature of the volume is the application of Reciprocity to many of the best known theorems by which the corresponding properties of the Conic are ascertained. This method and that of Inversion are pursued as far as is admissible within the scope and limits of an elementary treatise on Geometry.

In the preparation of the book, I consulted chiefly the writings of Mulcahy, Cremona, Catalan, Salmon, and Townsend, and hereby acknowledge my indebtedness for the valuable stores of information thus placed at my disposal.

Many of the Examples are from the Dublin University Examination Papers, and more especially from those set by Mr. M'Cay.

I have as far as possible indicated my additional sources of information, and given the reader references to the original memoirs from which extracts have been taken.

WILLIAM J. McCLELLAND.

SANTRY SCHOOL,  
1st November, 1891.

# CONTENTS.

## CHAPTER I.

### INTRODUCTION.

ARTICLE	PAGE
1. Symmetry, Convention of Positive and Negative, . . . . .	1
3. Anharmonic Ratios, . . . . .	3
<i>Examples</i> , . . . . .	4
Euler's Theorem, . . . . .	5
4. Limiting Cases, 0 and $\infty$ , . . . . .	6
<i>Examples</i> , . . . . .	7
5, 6. Envelopes, . . . . .	9
<i>Examples</i> . Bobillier's and Mannheim's Theorems, . . . . .	10
7. Casey's Theorem $23 \cdot 14 + 31 \cdot 24 + 12 \cdot 34 = 0$ , . . . . .	11
<i>Examples</i> . Feuerbach's Theorem, Hart's Extension, . . . . .	13

## CHAPTER II.

### MAXIMUM AND MINIMUM.

#### SECTION I.

##### INTRODUCTION.

8. Explanation of Terms, . . . . .	15
<i>Examples</i> , . . . . .	17
9. Theorem, . . . . .	17
<i>Examples</i> , . . . . .	18
10. Theorem, . . . . .	21
<i>Examples</i> , . . . . .	22
11. Problem, . . . . .	23
12. Theorem. Corollaries, . . . . .	23
<i>Examples</i> , . . . . .	26
13. Problem, . . . . .	26
<i>Examples</i> , . . . . .	27



ARTICLE	PAGE
14. Theorem. <i>Examples</i> , . . . . .	27
15. Theorem, . . . . .	30
16. Problem, . . . . .	30
<i>Examples</i> , . . . . .	31

## SECTION II.

## METHOD OF INFINITESIMALS.

17. 18. Explanatory, . . . . .	32
<i>Examples</i> , . . . . .	35

## SECTION III.

19. The Point $O$ Theorem, . . . . .	40
20. Similar Loci, . . . . .	42
<i>Examples</i> , . . . . .	43
21-23. Additional Theorems, . . . . .	44
<i>Examples</i> , . . . . .	45
Extension of Ptolemy's Theorem, . . . . .	48
24. Theorem, . . . . .	49
<i>Examples</i> , . . . . .	50
25. Theorem. Centre of Similitude, . . . . .	51
<i>Examples</i> , . . . . .	53

## SECTION IV.

26. MISCELLANEOUS PROPOSITIONS.	59
---------------------------------	----

## CHAPTER III.

## RECENT GEOMETRY.

## SECTION I.

## THE BROCARD POINTS AND CIRCLE OF A TRIANGLE.

27. Brocard Points ( $\Omega$ and $\Omega'$ ), . . . . .	60
28. Brocard Angle ( $\omega$ ), . . . . .	61
$\cot \omega = \cot A + \cot B + \cot C$ , . . . . .	61
<i>Examples</i> , . . . . .	62
Brocard Circle and Brocard's First Triangle, . . . . .	63
Polar Equation of a Circle, . . . . .	65

SECTION II.

THE SYMMEDIANS OF A TRIANGLE.

ARTICLE	PAGE
29. Fundamental Property of Symmedians, . . . . .	66
30. Expression for Length of Symmedian. <i>Examples</i> , . . . . .	67
31. Antiparallels, . . . . .	68
32. The Pedal Triangles of the Brocard Points, . . . . .	69
33. Theorems, . . . . .	70

SECTION III.

34. TUCKER'S CIRCLES. . . . .	71
35. Construction for Tucker's Circles, . . . . .	72
36-38. Theorems, . . . . .	73

SECTION IV.

TUCKER'S CIRCLES. PARTICULAR CASES.

39. Cosine Circle, . . . . .	74
40. Hain's Property of the Symmedian Point, . . . . .	75
42. Triplicate Ratio ("T. R.") Circle, . . . . .	75
45. Taylor's Circle, . . . . .	76
48. Common Orthogonal Circle of the three ex-Circles of a Triangle, . . . . .	78
<i>Examples</i> , . . . . .	79

CHAPTER IV.

GENERAL THEORY OF THE MEAN CENTRE OF  
A SYSTEM OF POINTS.

50. Theorem, . . . . .	84
51. Theorem, . . . . .	85
<i>Examples</i> , . . . . .	86
52. Theorem. <i>Examples</i> , . . . . .	87
Isogonal and Isotomic Conjugates, . . . . .	88
53. Method of finding the Mean Centre, . . . . .	90
Corollaries, . . . . .	91
<i>Examples</i> , . . . . .	92
Weill's Theorem, . . . . .	96
54, 55. Properties of Mean Centre. Corollaries, . . . . .	98
<i>Examples</i> , . . . . .	100

ARTICLE	PAGE
56. Theorem. Corollaries, . . . . .	103
<i>Examples</i> , . . . . .	104

## RECIPROCAL THEOREMS.

57. Central Axis, . . . . .	106
58. Method of finding Central Axis, . . . . .	107
60. Diameter of a Polygon, . . . . .	108
61. $\Sigma a \cdot AP^2$ a Minimum, . . . . .	108
<i>Examples</i> , . . . . .	109

## CHAPTER V.

## COLLINEAR POINTS AND CONCURRENT LINES.

62. Theorem $\frac{BX}{CX} \cdot \frac{CY}{AY} \cdot \frac{AZ}{BZ} = 1$ , . . . . .	112
63. Theorem $\frac{BX}{CX} \cdot \frac{CY}{AY} \cdot \frac{AZ}{BZ} = -1$ , . . . . .	113
64. Equivalent Relations. Particular Cases, . . . . .	114
<i>Examples</i> , . . . . .	117
Centre of Perspective, . . . . .	120
65. Criterion of Perspective of Two Triangles, . . . . .	120
66. Axis of Perspective. Homology, . . . . .	121
<i>Examples</i> , . . . . .	121
Stoll's Theorem, . . . . .	123
67. $\frac{BX \cdot BX'}{CX \cdot CX'} \cdot \frac{CY \cdot CY'}{AY \cdot AY'} \cdot \frac{AZ \cdot AZ'}{BZ \cdot BZ'} = 1$ , . . . . .	123
Corollaries. Pascal's Theorem, . . . . .	124
,, Brianchon's Theorem, . . . . .	125
,, Complete Cyclic Quadrilateral, . . . . .	127
<i>Examples</i> , . . . . .	128
Tarry's Point, . . . . .	131
68. Harmonic Properties of the Quadrilateral: Its Diagonal Triangle, . . . . .	132
Problem, . . . . .	133
<i>Examples</i> . Maximum Polygon of any Order escribed to a Circle, . . . . .	134
Note on Pascal and Brianchon Hexagons, . . . . .	136

CHAPTER VI.

INVERSE POINTS WITH RESPECT TO A CIRCLE.

ARTICLE	PAGE
69. Particular Cases, . . . . .	139
70, 71. Theorems, . . . . .	139
<i>Examples</i> , . . . . .	140
72. Theorem. Corollaries, . . . . .	140
<i>Examples</i> . Radical Centre and Product, Conjugate Lines of a Quadrilateral, Circle of Apollonius, . . . . .	143

CHAPTER VII.

POLES AND POLARS WITH RESPECT TO A CIRCLE.

SECTION I.

CONJUGATE POINTS. POLAR CIRCLE.

73. Definitions. Elementary Properties of Conjugate Points, . . . . .	149
74. Theorem, . . . . .	149
<i>Examples</i> , . . . . .	150
75, 76. Theorems. Reciprocal Polar Triangles, . . . . .	151
77. Self-Conjugate Triangles. Polar Circle, . . . . .	151
78. Expressions for the Radius of Polar Circle. Orthocentric System of Points, . . . . .	152
<i>Examples</i> . Polar Centre, . . . . .	153

SECTION II.

79. SALMON'S THEOREM. . . . .	155
Corollaries, . . . . .	156
<i>Examples</i> , . . . . .	157
Tangential Equations of In- and Circum-Circles, . . . . .	159

SECTION III.

RECIPROCATION.

80. Reciprocal Polar Curves. Their Elementary Properties, . . . . .	162
81. Reciprocation of a Circle from any Origin, . . . . .	164
<i>Examples</i> , . . . . .	167

## CHAPTER VIII.

## COAXAL CIRCLES.

## SECTION I.

## COAXAL CIRCLES.

ARTICLE	PAGE
82. Definitions, Radical Axis, . . . . .	173
83. Conjugate Coaxal Systems, . . . . .	173
Circular Points at Infinity, . . . . .	174
84. Modulus of a Coaxal System, . . . . .	174
85. To construct a Circle of given radius of a Coaxal System, .	175
86. Theorem, . . . . .	175
87. Limiting Points. Extreme Cases, . . . . .	175
88. Theorems. Corollaries, . . . . .	176
<i>Examples</i> , . . . . .	178
M'Cay's proof of Feuerbach's Theorem, . . . . .	183
89. Theorem, Corollaries, Centres and Circle of Similitude, .	184
<i>Examples</i> , . . . . .	186
Poncelet's Theorem, . . . . .	187
91. Theorems. Diagonal Line of a Quadrilateral, . . . . .	189
<i>Examples</i> , . . . . .	190

## SECTION II.

## ADDITIONAL CRITERIA OF COAXAL CIRCLES.

92. First Criterion. Corollaries, . . . . .	191
<i>Examples</i> , . . . . .	192
93. Second Criterion. Corollaries, . . . . .	194
<i>Examples</i> , . . . . .	195

## SECTION III.

## CIRCLE OF SIMILITUDE.

94, 95. Fundamental Properties, . . . . .	197
<i>Examples</i> , . . . . .	198
Orthocentroidal Circle, . . . . .	200
Miscellaneous Examples, . . . . .	200
Extension of Weill's Theorem, . . . . .	201

# CHAPTER IX.

## THEORY OF SIMILAR FIGURES.

## SECTION I.

## TWO SIMILAR FIGURES.

ARTICLE	PAGE
96. Homothetic Figures, . . . . .	204
97. Fundamental Properties. Homothetic Centre, Centre of Similitude, . . . . .	205
98. Double Points of Similar Polygons, . . . . .	205
99. Further Properties, Instantaneous Centre, . . . . .	206

## SECTION II.

## THREE SIMILAR FIGURES.

100. Triangle of Similitude, . . . . .	207
Circle of Similitude, Invariable Points. Invariable Triangle, . . . . .	208
101, 102. Theorems, . . . . .	209
103. Adjoint Points, Director Point, . . . . .	209
104. Theorems by Neuberg, . . . . .	210
105. Particular Cases, . . . . .	210
106. M'CAY'S CIRCLES. . . . .	211
107. To find Centres and Radii of M'Cay's Circles, . . . . .	213
Examples, . . . . .	216
Feuerbach's Theorem, . . . . .	217

## CHAPTER X.

## CIRCLES OF SIMILITUDE AND OF ANTISIMILITUDE.

## SECTION I.

109. CENTRES OF SIMILITUDE. . . . .	219
110. Properties of Homologous and Antihomologous Points. . . . .	220
111. Theorem. Homologous Chords are Parallel, . . . . .	221
112. Theorem. Antihomologous Chords meet on their Radical Axis. . . . .	221
113. Products of Antisimilitude. Their Values. Note, . . . . .	222
Examples, . . . . .	223
Feuerbach's Theorem, . . . . .	225

## SECTION II.

## CIRCLES OF ANTISIMILITUDE.

ARTICLE	PAGE
Definitions, . . . . .	227
114. Fundamental Properties, . . . . .	227
<i>Examples</i> , . . . . .	229
Constructions for the eight circles touching three given ones, . . . . .	230

## CHAPTER XI.

## INVERSION.

## SECTION I.

## INTRODUCTORY.

115. Definitions, . . . . .	234
116. Species of Triangle obtained by Inversion of the Vertices of a given one, . . . . .	236
118. Relations between the Sides of the two, . . . . .	237
119. Theorem. Corollaries, . . . . .	237
120. Inversion of a System of Four Points. Particular Cases. Note, . . . . .	238
121. Relations between the sides of $ABCD$ and $A'B'C'D'$ . Corollaries. Inversion of the Vertices of an Harmonic Quadrilateral. . . . .	239
<i>Examples</i> , . . . . .	240
122. Theorem, . . . . .	241
<i>Examples</i> , . . . . .	242

## SECTION II.

## ANGLES OF INTERSECTION OF FIGURES AND OF THEIR INVERSES.

123. General Relations between a Circle and its Inverse from any Origin, . . . . .	243
124. Problem. Corollary, . . . . .	245
<i>Examples</i> , . . . . .	246
125. Theorem, . . . . .	246
126. Angle between two Curves remains unaltered by Inversion, .	247
127. Important Considerations arising from the Theorem of Art. 126, . . . . .	248

## CHAPTER I.

### INTRODUCTION.

**Definitions.**—Right lines passing through a point are called a *Concurrent System*.

The point is the *Vertex* of the system, and the lines are a *Pencil of Rays*.

*Collinear points* are those which lie on a right line.

### **Symmetry. Convention of Positive and Negative.**—

1. The letters  $A, B, C, \dots$ , are generally used to denote points and positions of lines, and  $a, b, c$ , *lengths*, e.g., the vertices of a triangle are  $A, B, C$ , and the opposite sides  $a, b, c$ .

By  $AB$  is meant the distance from  $A$  to  $B$  measured from  $A$  towards  $B$ , and by  $BA$  the same distance measured in the *opposite* direction.

Thus  $AB = -BA$  or  $AB + BA = 0$ .

Similarly for three collinear points  $A, B, C$ :

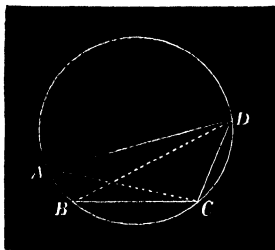
$AB + BC = AC = -CA$ , therefore  $BC + CA + AB = 0$ .

2. If four points  $A, B, C, D$ , be taken in alphabetical order on a circle, we have by Ptolemy's Theorem

$$BC \cdot AD + AB \cdot CD = BD \cdot AC = -CA \cdot BD,$$



the six linear segments being measured from left to right, or we shall say *positively*, in figure ;



hence, by transposing,

$$BC \cdot AD + CA \cdot BD + AB \cdot CD = 0.$$

Again, since each chord is proportional to the sine of the angle it subtends at any fifth point  $O$  on the circle, this equation reduces to

$$\sin BOC \sin AOD + \sin COA \sin BOD + \sin AOB \sin COD = 0,$$

a result which is therefore true for any pencil of four lines, and is deduced directly from Ptolemy's Theorem by describing a circle of any radius through its vertex.

In this equation it is implied that  $\angle AOC$  denotes the magnitude of the angle measured from  $A$  towards  $C$ , and that therefore  $\sin AOC = -\sin COA$ .

3. Let  $O.ABCD$  denote a system of lines concurrent at  $O$ ;  $A, B, C, D$ , the points in which a line  $L$  meets it; and  $p$  the distance of the vertex  $O$  from  $L$ .

Then  $2BOC = BC \cdot p = OB \cdot OC \sin BOC$ ,  
and  $2AOD = AD \cdot p = OA \cdot OD \sin AOD$ ;

by multiplication

$$BC \cdot AD \cdot p^2 = OA \cdot OB \cdot OC \cdot OD \cdot \sin BOC \cdot \sin AOD; \dots (1)$$

similarly

$$CA \cdot BD \cdot p^3 = OA \cdot OB \cdot OC \cdot OD \cdot \sin COA \cdot \sin BOD; \quad (2)$$

dividing (1) by (2) we have

$$BC \cdot AD : CA \cdot BD = \sin BOC \cdot \sin AOD : \sin COA \cdot \sin BOD. \quad (3)$$

The student will observe that three pairs of angles are formed by taking any pair of rays with the remaining or *Conjugate* pair.

Thus  $BOC$  and  $AOD$  may be conveniently denoted by  $\alpha$  and  $\alpha'$ ,  $COA$  and  $BOD$  by  $\beta$  and  $\beta'$ , and  $AOB$  and  $COD$  by  $\gamma$  and  $\gamma'$ .

With this notation (3) is written

$$BC \cdot AD : CA \cdot BD = \sin \alpha \sin \alpha' : \sin \beta \sin \beta',$$

and generally we infer from symmetry that

$$BC \cdot AD : CA \cdot BD : AB \cdot CD = \sin \alpha \sin \alpha' : \sin \beta \sin \beta' : \sin \gamma \sin \gamma'. \quad (4)$$

COR. 1. If we draw four parallels to the rays of the pencil, we in general obtain a triangle and a transversal to its sides. Moreover, if we denote the angles of the triangle by  $\alpha, \beta, \gamma$ , those made by the transversal with its sides are the opposites  $\alpha', \beta', \gamma'$ ; hence for any triangle and transversal we have always

$$\sin \alpha \sin \alpha' + \sin \beta \sin \beta' + \sin \gamma \sin \gamma' = 0.$$

COR. 2. Let the line  $ABCD$  be divided *harmonically* or such that  $AB/BC = AD/CD$ , then  $BC \cdot AD = AB \cdot CD$ ; hence by (3) the pencil is divided harmonically, *i.e.*, the angle  $COA$  is divided internally in  $B$  and externally in  $D$  in the same ratio of sines.

**Defs.** The three ratios and their reciprocals on the left side of (3) are termed the *Anharmonic Ratios* of the four points on the line; and those on the right the *Anharmonic Ratios* of the pencil  $O \cdot ABCD$ .

Their equivalence is expressed thus:—A variable line drawn across a pencil is cut in a constant anharmonic ratio; or any pencil and transversal to it are *Equianharmonic*.

The foot of the perpendicular from a point on a line is the *Projection of the point* on the line, and the perpendicular is called its *Projector*.

If  $A'$  and  $B'$  be the projections of  $A$  and  $B$  on a line  $L$ ,  $A'B'$  is called the *Projection of  $AB$* , and is equal to  $AB \cos \theta$ , where  $\theta$  is the angle between  $AB$  and  $L$ .

#### EXAMPLES.

1. The sum of the projections of the sides of a polygon on any right line = 0; and generally if lines be drawn equally inclined and proportional to the sides of a polygon, the sum of their projections is zero.

$$2. \quad \cos a + \cos\left(a + \frac{2\pi}{n}\right) + \cos\left(a + \frac{4\pi}{n}\right) + \dots + \cos\left(a + \frac{2(n-1)\pi}{n}\right) = 0,$$

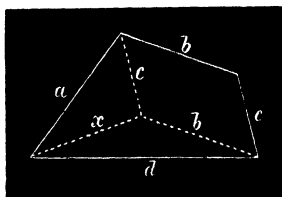
and the sum of the sines of the series of angles is also equal to 0.

[For they are proportional to the projections of the sides of a regular polygon on two lines at right angles.]

3. In any quadrilateral whose sides are  $a, b, c, d$ , to prove that

$$d^2 = a^2 + b^2 + c^2 - 2bc \cos \hat{bc} - 2ca \cos \hat{ca} - 2ab \cos \hat{ab},$$

where  $\hat{bc}$  denotes the angle between the sides  $b$  and  $c$ .



[For completing the parallelogram whose sides are  $b$  and  $c$  and drawing  $x$  we have  $d^2 = b^2 + c^2 + 2bc \cos x$ ,

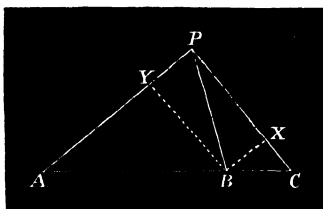
where  $x'$  is the projection of  $x$  on the parallel  $b$ ; but by Ex. 1.

$$x' = a \cos \hat{ab} + c \cos \hat{bc},$$

substituting for  $x'$  its value and for  $x^2$ ,  $a^2 + c^2 - 2ac \cos \hat{ac}$ , the above result is obtained.]

4. **Euler's Theorem.\***—For three collinear points  $A$ ,  $B$ ,  $C$  and any fourth  $P$  to prove the relation

$$BC \cdot AP^2 + CA \cdot BP^2 + AB \cdot CP^2 = -BC \cdot CA \cdot AB.$$



[By Euc. II. 12, 13,  $AP^2 = AB^2 + BP^2 - 2AB \cdot BP \cos B$ .....(1)  
and  $CP^2 = BC^2 + BP^2 + 2BC \cdot BP \cos B$ .....(2)

multiplying (1) by  $BC$  and (2) by  $AB$  and adding to eliminate  $\cos B$ , the above follows on reduction.]

4A. Having given the base  $c$  of a triangle and  $la^2 + mb^2 = \text{const.}$ , find the locus of the vertex,  $l$  and  $m$  being given quantities.

5. If  $APC$  is a right angle the relation in Ex. 4 is equivalent to

$$BC^2 \cdot AP^2 + AB^2 \cdot CP^2 = AC^2 \cdot BP^2.$$

[This follows from Ex. 4 or is obtained directly thus; let fall perpendiculars  $BX$  and  $BY$  on  $CP$  and  $AP$ , then

$$XY^2 = BP^2 = BX^2 + BY^2 = BC^2 \sin^2 C + AB^2 \sin^2 A;$$

multiplying the equation  $BP^2 = BC^2 \sin^2 C + AB^2 \sin^2 A$  by  $AC^2$ ; therefore, etc.].

6. If the transversal to a harmonic pencil is parallel to one ray  $D$ , the intercept  $AC$  is bisected by  $B$  the conjugate of  $D$ .

\* "Catalan's *Théorèmes et Problèmes de Géométrie Élémentaire*," 1879, p. 141.

7. If a line  $L$  turn around a fixed point  $P$  and meet two fixed lines  $OA$  and  $OB$  in  $A'$  and  $B'$ ; the locus of the harmonic conjugate  $Q$  of  $P$  with respect to  $A'B'$  is a line passing through  $O$ ; and

$$\frac{1}{PA'} + \frac{1}{PB'} = \frac{2}{PQ} \dots\dots\dots (\text{By Ex. 6.})$$

Note. By Euc. VI. 2 if the variable  $PQ$  is bisected at  $Q'$  the locus of  $Q'$  is a parallel to  $OQ$  and

$$\frac{1}{PA'} + \frac{1}{PB'} = \frac{1}{PQ'}$$

Hence for any three lines  $A, B, C$  we find in the same manner that

$$\frac{1}{PA'} + \frac{1}{PB'} + \frac{1}{PC'} = \frac{1}{PQ'}$$

where  $Q'$  describes a right line.

8. For any system of lines  $A, B, C, D \dots$  the locus of  $Q'$  such that

$$\frac{1}{PA'} + \frac{1}{PB'} + \frac{1}{PC'} + \dots = \frac{1}{PQ'} \quad \left( \text{or } \sum \frac{1}{PA'} = \frac{1}{PQ'} \right)$$

is a right line. [See Exs. 6 and 7.]

9. For a regular cyclic polygon, if  $P$  coincides with the centre

$$\sum \frac{1}{PA'} = 0.$$

[Through  $P$  draw the line parallel to one of the sides, etc.]

10. If parallels be drawn through any point  $O$  to the four lines in Ex. 4, the relation may be written

$$\frac{\sin \beta' \sin \gamma'}{\sin \beta \sin \gamma} + \frac{\sin \gamma' \sin \alpha'}{\sin \gamma \sin \alpha} + \frac{\sin \alpha' \sin \beta'}{\sin \alpha \sin \beta} = 1.$$

11. From the formula  $BC \cdot AD + CA \cdot BD + AB \cdot CD = 0$ , prove that if  $A, B, C$  be three collinear points and  $P$  any fourth point  $BC \cot A + CA \cot B + AB \cot C = 0$ , the angles being all measured in the same aspect; and hence find the locus of the vertex, having given the base  $c$  and  $l \cot A + m \cot B = \text{const.}$

#### 4. Limiting Cases. 0 and $\infty$ .

**Def.** The *Angle of intersection* of two circles is that between the tangents drawn to them at either point of

intersection; it is therefore equal to the angle between the radii drawn to either common point.\* (Euc. III. 19.)

If the circles touch *Internally* this angle is  $0^\circ$ , if *Externally*  $180^\circ$ . They are said to intersect *Orthogonally* when the angle is  $90^\circ$ .

The *Angle made by a line and circle* is that between the line and the tangent to the circle at its intersection.

## EXAMPLES.

1. To find the angles between the circum- and ex-circles of a triangle  $ABC$ .

[Since  $\delta_1^2 = R^2 + 2Rr_1$ , etc., we easily obtain  $2\cos\frac{1}{2}\theta_1 = \sqrt{\frac{r_1}{R}}$ ; with similar expressions for  $\theta_2$  and  $\theta_3$ .]

2. To find the angle of intersection of the in- and circum-circles.

[ $\delta^2 = R^2 - 2Rr$ , therefore  $2\sin\frac{1}{2}\theta = \sqrt{\frac{ir}{R}}$  where  $\sqrt{-1} = i$ .]

3. If two concentric circles cut orthogonally one is real and the other imaginary, and their radii are of the forms  $\rho, i\rho$ .

\* If  $O_1, r_1; O_2, r_2$ , be the circles,  $\delta$  the distance  $O_1O_2$ ,  $\theta$  the angle of intersection, and  $t$  the direct common tangent, we have

$$\begin{aligned}\delta^2 &= r_1^2 + r_2^2 - 2r_1r_2 \cos \theta \\ &= (r_1 - r_2)^2 + 4r_1r_2 \sin^2 \frac{1}{2}\theta;\end{aligned}$$

hence  $\delta^2 - (r_1 - r_2)^2 = 4r_1r_2 \sin^2 \frac{1}{2}\theta$ ,.....(1)

or  $\frac{t^2}{4r_1r_2} = \sin^2 \frac{1}{2}\theta$ .

Similarly  $\delta^2 - (r_1 + r_2)^2 = -4r_1r_2 \cos^2 \frac{1}{2}\theta$ ,

hence if  $t'$  be the transverse common tangent,

$$t'^2 = -4r_1r_2 \cos^2 \frac{1}{2}\theta$$
.....(2)

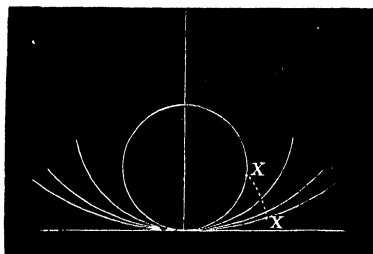
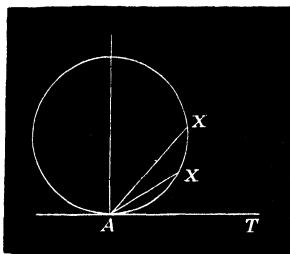
Multiplying (1) and (2) and reducing we have

$$tt' = 2i \cdot r_1r_2 \sin \theta$$
.....(3)

where  $\sqrt{-1} = i$ ; also if  $\gamma$  denote the length of the common chord, of the circles (real or imaginary) since  $2r_1r_2 \sin \theta = \gamma\delta$ ,  $t \cdot t' = i \cdot \gamma \cdot \delta$ .

It is obvious that either the transverse common tangent to the circles or their angle of intersection is imaginary.

Let  $AX$  be a variable chord passing through a fixed point  $A$  at which a tangent is drawn. According as the



chord  $AX$  and angle  $TAX$  diminish in magnitude  $X$  approaches the tangent. When  $X$  is indefinitely near to  $A$ ,  $AX$  is said to have reached its *limiting position* and may then be considered to coincide with the tangent.

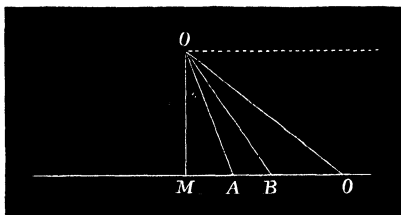
Hence a *tangent to a circle is in the direction of the infinitesimal chord at its point of contact, or is the chord joining two indefinitely near points.*

Again, let the tangent  $T$  and its point of contact be fixed and the chord  $AX$  given in length. As the radius of the circle increases the curvature diminishes, and the point  $X$  obviously approaches the tangent. Hence  $X$  may be made to move as near as we please to the tangent by continually increasing the value of the radius of the circle.

In the limit, when the latter is indefinitely great, the distance of  $X$  from  $T$  is so very small that we may consider the point to lie on the line. Hence a finite portion of a circle of indefinitely great radius opens out into a right line, the remainder being, of course, at a distance infinitely great, i.e., at infinity.

5. **Envelopes.**—Let a variable line turn around a fixed point  $O$  and meet any fixed line.

According as its angle of inclination to the perpendicular  $OM$  increases, the segments  $OA$ ,  $OB$ ,  $OC$  continue to increase and the angles  $A$ ,  $B$ ,  $C$  to diminish. In the



limit it reaches a position at right angles to  $OM$ . Here the angle between it and the fixed line vanishes, and their point of section is at infinity. In this case the lines are parallel (Euc. I. 28); hence

*Parallel lines may be regarded as having angles of inclination =  $0^\circ$  or lines intersecting at infinity.* Thus a system of parallels is a pencil of rays whose vertex is at infinity.

6. Let  $A$  and  $X$  be any two points on a curve of which  $A$  is fixed and  $X$  variable, and  $TA$  and  $TX$  tangents. It appears as before that as  $X$  approaches  $A$  the chord  $AX$  and the base angles  $A$  and  $X$  of the triangle  $TAX$  gradually diminish and ultimately vanish.

But as the base angles diminish the vertex  $T$  approaches the base and *a fortiori* the element of curve  $AX$ . Hence in the limiting position, *i.e.*, when the tangents are consecutive, their point of intersection is on the curve.

A curve touched by a variable line is called the *Envelope* of the line. Thus the envelope of a line which



*varies according to any law is the locus of the intersection of its consecutive positions.*

### EXAMPLES.

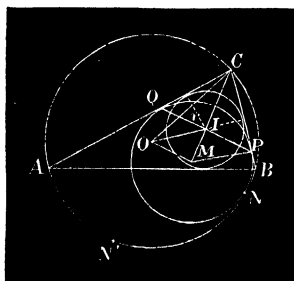
1. The envelope of equal chords in a circle is a concentric circle (Euc. III. 14).

2. **Bobillier's Theorem.**—If two sides of a given triangle touch fixed circles the third side also touches, or envelopes, a circle.

[Let  $ABC$  be the given triangle. Through  $O_1$  and  $O_2$ , the centres of the given circles, draw parallels to the sides meeting the base in  $A'$  and  $B'$  and each other in  $C'$ . Describe a circle  $O_1O_2C'$ , and draw  $C'O_3$  parallel to  $AB$ .

Since  $O_2C'O_3$  is a given angle ( $=A$ ),  $O_3$  is a fixed point. But  $A'B'C'$  is given in all respects save position; hence the distance  $p$  of  $O_3$  from  $A'B'$  is a known quantity. The envelope of the base  $AB$  is therefore a circle whose centre is  $O_3$  and radius  $=p$ .]

3. To find the radius ( $\rho$ ) of a circle which touches the sides  $AC$  and  $BC$  of a triangle and the circum-circle  $ABC$ .



[Let  $I$  denote the in- and  $O$  the circum-centre of the triangle;  $M$  the centre of the circle whose radius is required is on the line  $CI$ . Then  $OM = R - \rho$ ,  $OI^2 = R^2 - 2Rr$ ,  $IC = r/\sin \frac{1}{2}C$ ,  $MC = \rho/\sin \frac{1}{2}C$ , and  $MI = (\rho - r)/\sin \frac{1}{2}C$ .

Also, since  $C, I, M$  are three points in a line and  $O$  any fourth point, by *Euler's Theorem* we obtain on reducing

$$r = \rho \cos^2 \frac{1}{2}C \dots \dots \dots (1)$$

Again, if the circle  $M, \rho$  has external contact with the circum-circle, it can be similarly proved that

$$r_3 = \rho \cos^2 \frac{1}{2} C \dots \dots \dots (2)$$

NOTE.—The relation (1) is otherwise expressed :—

Since  $r/\rho = \cos^2 \frac{1}{2} C$ ,  $(\rho - r)/\rho = \sin^2 \frac{1}{2} C$ .

But  $(\rho - r)/\rho = MI/MC$  and  $\rho^2/MC^2 = \sin^2 \frac{1}{2} C$ ,

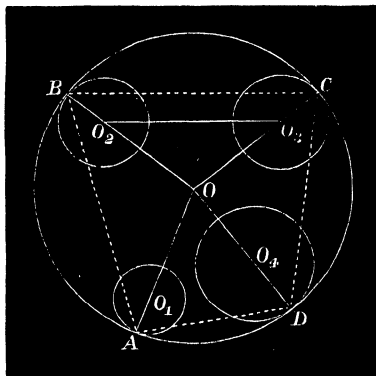
hence  $MI \cdot MC = \rho^2 \dots \dots \dots (3)$

or the chord of contact  $PQ$  of the circle  $M, \rho$  with the sides of the triangle passes through the centre of the inscribed circle.]

4. **Mannheim's Theorem.**—Having given the vertical angle and radius of the in- or corresponding ex-circle, the envelope of the circum-circle is a circle.

[By Ex. 3.]

7. We shall conclude the present chapter with the following useful property, of the common tangents to four circles which touch a fifth, due to the late Dr. Casey.



Denote the circle whose centre is  $O$  and radius  $r$  by  $O, r$ ; and let the four circles  $O_1 r_1, O_2 r_2, O_3 r_3, O_4 r_4$ , touch a fifth  $O, R$  at the points  $A, B, C, D$ . Let the distance  $O_2 O_3$  be  $\delta_{23}$ , and the direct common tangent to the corresponding circles be  $\overline{23}$ .

Then  $\overline{23}^2 = \delta_{23}^2 - (r_2 - r_3)^2$ .

In the triangle  $OO_2O_3$  we have

$$\begin{aligned} OO_2^2 &= OO_3^2 + OO_2^2 - 2OO_2 \cdot OO_3 \cos BOC \\ &= (OO_2 - OO_3)^2 + 4OO_2 \cdot OO_3 \sin^2 \frac{1}{2} BOC; \end{aligned}$$

$$\text{or} \quad \delta_{23}^2 - (r_2 - r_3)^2 = 4OO_2 \cdot OO_3 \sin^2 \frac{1}{2} BOC;$$

$$\text{or} \quad \frac{\delta_{23}^2}{23^2} = 4OO_2 \cdot OO_3 \sin^2 \frac{1}{2} BOC = OO_2 \cdot OO_3 \cdot BC^2/R^2.$$

Similarly

$$\overline{14}^2 = OO_1 \cdot OO_4 \cdot AD^2/R^2;$$

hence by multiplication and reduction

$$\overline{23} \cdot \overline{14} = (OO_1 \cdot OO_2 \cdot OO_3 \cdot OO_4)^{\frac{1}{2}} BC \cdot AD/R^2,$$

and by Ptolemy's Theorem

$$\overline{23} \cdot \overline{14} + \overline{31} \cdot \overline{24} + \overline{12} \cdot \overline{34} = 0 \dots \dots \dots (1).$$

The contacts in the figure are similar, or all of the same kind, but it will be observed that if the fifth circle touches any two with contacts of opposite species, their transverse common tangents must be substituted in (1).

We let  $\overline{12}'$  denote the transverse tangent to  $O_1, r_1$  and  $O_2, r_2$ ; then

$$\overline{12}'^2 = \delta_{12}^2 - (r_1 + r_2)^2.$$

For example, if the circle  $O_1, r_1$  is external and the remaining circles internal to  $O, R$  the relation is written

$$\overline{23} \cdot \overline{14}' + \overline{31}' \cdot \overline{24} + \overline{12}' \cdot \overline{34} = 0,$$

with analogous expressions for all other cases.

NOTE.—The student must carefully observe that of the three terms of the equation two are positive and one negative; the latter corresponding to the pairs of circles whose contacts are alternate. Thus in the figure,  $O_1, r_1$  and  $O_3, r_3$  have alternate contacts with the given circle, therefore the term  $\overline{31} \cdot \overline{24}$  is negative, and taking the absolute values only the equation is

$$\overline{23} \cdot \overline{14} + \overline{12} \cdot \overline{34} = \overline{31} \cdot \overline{24}.$$

This is of great importance, and should be borne in mind in the following Examples.

## EXAMPLES.

1. What does the general property reduce to when the circles become points? Ptolemy's Theorem.

2. Express a condition that the circum-circle of a given triangle may touch another circle.

[If  $a, b, c$  be the sides and  $t_1, t_2, t_3$  the tangents from the vertices to the other circle we have  $at_1 + bt_2 + ct_3 = 0$ .]

3. **Feuerbach's Theorem.**—The nine points circle of a triangle touches the in- and ex-circles.

[The middle points of the sides and the in-circle are four circles satisfying the equation of Ex. 2. For  $2\bar{3} = \frac{1}{2}a$  and  $1\bar{4} = \frac{1}{2}(b - c)$ ; therefore  $\Sigma 2\bar{3} \cdot 1\bar{4} = \frac{1}{4}\Sigma a(b - c) = 0^*$ .]

4. If  $a, b, c$  be the sides of a triangle inscribed in a circle, and  $\lambda, \mu, \nu$  the distances of its vertices from any tangent, show that the equation in Ex. 2 reduces to

$$a\sqrt{\lambda} + b\sqrt{\mu} + c\sqrt{\nu} = 0.†$$

5. More generally if  $\lambda, \mu, \nu$  denote the distances from any line, give the geometrical interpretation of the equation

$$a\sqrt{\lambda - x} + b\sqrt{\mu - x} + c\sqrt{\nu - x} = 0,$$

and hence find a relation connecting the sides of a triangle with the distances of its vertices from a given line.

[The roots of the quadratic in  $x$  are the distances from the line of the tangents to the circle parallel to it, etc.]

6. **Hart's Extension of Feuerbach's Theorem.**—If the sides of a triangle be replaced by three circles, and four circles corresponding to the in- and ex-circles of the triangle described to touch them; the group of four is touched by a circle.

[Let the triangle formed by the circles be  $ABC$ , and let  $a < b < c$ . Then  $s - a > s - b > s - c$ . If the in- and ex-circles are numbered

\* This proof is an application of the *converse* of Dr. Casey's relation.

† This result may be otherwise shown as follows:—Let  $P$  be the point of contact of the tangent. Then  $BC \cdot AP + CA \cdot BP + AB \cdot CP = 0$ . But  $AP^2 = 2r\lambda$ ,  $BP^2 = 2r\mu$ , and  $CP^2 = 2r\nu$ , substituting these values; therefore, etc.

1, 2, 3, 4 respectively, the side  $a$  is touched by the four circles and the transverse tangents are drawn to 2 ; also the order of the contacts is 3, 1, 2, 4 ; hence the equation is

$$-\overline{23'} \cdot \overline{14} + \overline{31} \cdot \overline{24'} + \overline{12'} \cdot \overline{34} = 0 \dots \dots \dots (1)$$

For the side  $b$  the transverse tangents are drawn to 3, and the order of the contacts is 2, 1, 3, 4 ; hence

$$-\overline{23'} \cdot \overline{14} + \overline{31'} \cdot \overline{24} + \overline{12} \cdot \overline{34'} = 0 \dots \dots \dots (2)$$

For the side  $c$  the transverse tangents are drawn to 4, and the order of the contacts is 3, 4, 1, 2 ; hence

$$\overline{23} \cdot \overline{14'} - \overline{31} \cdot \overline{24'} + \overline{12} \cdot \overline{34'} = 0 \dots \dots \dots (3)$$

Adding (1) and (3) and subtracting (2) we get

$$\overline{23} \cdot \overline{14'} - \overline{31'} \cdot \overline{24} + \overline{12'} \cdot \overline{34} = 0,$$

showing that 2, 3, 4 have similar and 1 opposite contacts with a circle which touches all four.]

## CHAPTER II.

### MAXIMUM AND MINIMUM—INTRODUCTION.

8. When the base and vertical angle of a triangle are given the locus of the vertex is a segment of a circle described on the base, containing an angle equal to the vertical angle. (Euc. III. 21.) Let a number of triangles be constructed satisfying the given conditions, and it will be observed that as the vertex recedes from either extremity of the base the altitude and area both increase up to a certain point, after which they begin to diminish.

This point is obviously the middle point of the segment—the vertex of the isosceles triangle with the given parts—or the point at which the tangent to the arc is parallel to the base.

Here the area and altitude are said to have attained their *maximum* values.

Again since the rectangle under the sides  $AC$  and  $BC$  is equal to the rectangle under the diameter of the circum-circle and altitude ( $ab = dp$ );  $ab$  and  $p$  are maxima simultaneously.

Also since  $a^2 + b^2 = 2(\frac{1}{2}c)^2 + 2\beta^2$ ,

where  $\beta$  is the median to the side  $c$ ; when  $\beta$  is a maximum or minimum,  $a^2 + b^2$  is maximum or minimum.

And if  $N$  be the middle point of the arc of the circle below the base, then, since  $AN = BN$  ( $=x$  say) by Ptolemy's Theorem, we have

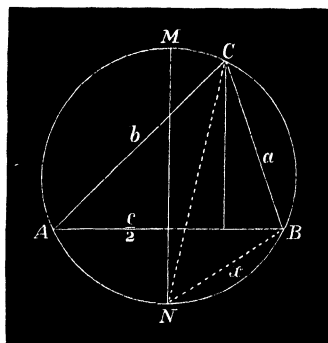
$$ax + bx = c \cdot CN,$$

or

$$x(a + b) = c \cdot CN,$$

from which it appears that  $a + b$  and  $CN$  are maxima together; that is when the vertex  $C$  is at the middle point  $M$  of the arc  $AB$ .

On the other hand it is manifest that the difference of base angles ( $A - B$ ) and difference of sides ( $a - b$ ) both diminish as the vertex  $C$  approaches  $M$  and vanish at that point; and after  $C$  passes through this point each difference begins to increase. At  $C$  they are said to have their *minimum* values, though this need not necessarily be nothing.



Thus generally:—a variable quantity which, under certain conditions, increases up to a definite limit and then begins to diminish, is said to have attained its maximum value at the limit; and if, after diminishing, it again begins to increase, it attains a minimum value at the stage where it has ceased to diminish.

The foregoing remarks may be thus summed up:—Of all triangles having a given base and vertical angle the isosceles has the following maxima—area, altitude, rectangle under sides, sum of sides, bisector of base, and sum of squares of sides.\*

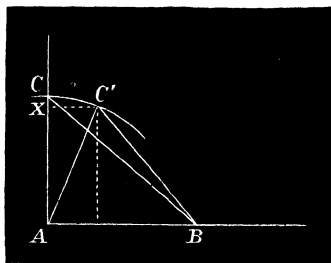
EXAMPLES.

1. The triangle of greatest area and perimeter inscribed in a circle is equilateral.

[For each vertex must lie mid-way between the other two, or the area and perimeter would both be increased by removing any vertex to the middle point.]

2. A regular polygon of  $n$  sides inscribed in a circle has a greater area and perimeter than any other inscribed polygon of the same order. [By Ex. 1.]

9. **Theorem.**—*If two sides  $AC$  and  $AB$  of a triangle are given in length the area of the triangle  $ABC$  is a maximum when they contain a right angle.*



Let  $ABC$  denote the right-angled triangle, and  $ABC'$  any other triangle formed with the given sides. Draw  $C'X$  perpendicular to  $AC$ .

Since  $AC = AC'$  and  $AC' > AX$ ; therefore  $AC > AX$ , hence (Euc. I. 41) the triangle  $ABC > ABC'$ , and similarly for any other position; therefore, etc.

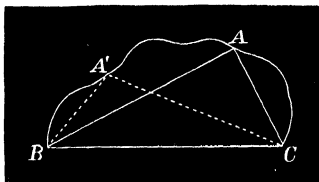
---

\* The vertical angle is supposed to be acute.



## EXAMPLES.

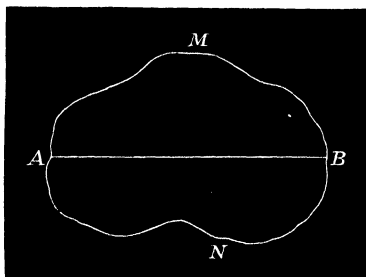
1. If the ends of a string of given length are joined, the area of the figure enclosed is a maximum when it takes the form of a semi-circle.



[Take any point  $A$  on the string  $ABC$  and join  $AB$  and  $AC$ . Consider the segments into which the string is divided at  $A$  to be rigidly attached to the lines  $AB$  and  $AC$ . If the angle at  $A$  is not right, by rotating  $AC$  around  $A$  until it is perpendicular to  $AB$ , the area of the triangle  $ABC$ , and therefore also of the whole figure, is increased.

Similarly for any other point  $A'$ ; hence the area enclosed is a maximum when the joining line  $BC$  subtends a right angle at every point on the string.]

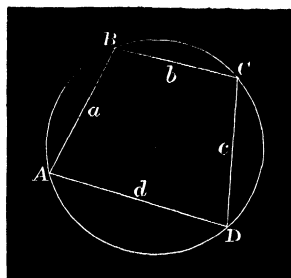
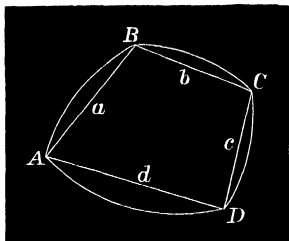
2. A closed curve of given perimeter is of greatest area when its form is a circle.



[Let  $A$  be any point on the curve, and take  $B$  such that  $AMB$  and  $ANB$  are equal in length. Then the areas  $AMB$  and  $ANB$  are

each a maximum when  $AB$  is the diameter of semicircles on opposite sides ; therefore, etc.]

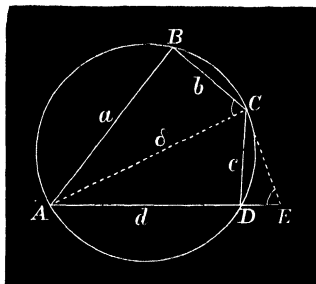
3. Having given the four sides  $a, b, c, d$  of a quadrilateral, its area is a maximum when it is cyclic.



[Let  $ABCD$  be the cyclic quadrilateral with the given sides, and consider the segments on the sides to be rigidly attached to them.\* If then the figure be distorted in any way into a new position

---

\* The construction of the cyclic quadrilateral whose four sides are given is as follows :—



Draw  $CE$  making  $\angle DCE = \angle BAC$ . Since by Euc. iii., 22,  $\angle CDE = \angle ABC$ , the triangles  $ABC$  and  $CDE$  are similar ; therefore  $DE : c = b : a$  (Euc. vi. 4) ; hence  $DE$  is known and  $E$  is a fixed point.

Again,  $AC : CE = a : c$  ; therefore in the triangle  $ACE$  we have the base  $AE$  and ratio of sides ; the locus of  $C$  is therefore a circle (Euc. vi. 3) ; this locus intersects the circle described with  $D$  as centre and  $c$  as radius at the point  $C$  ; therefore, etc.

$A'B'C'D'$ , the area of the circle  $ABCD > A'B'C'D'$  (Ex. 2), but the segments  $AB=A'B'$ ,  $BC=B'C'$ , etc.: take away these equal parts and there remains the quadrilateral  $ABCD$  greater than  $A'B'C'D'$ .]\*

4. If three sides  $a, b, c$ , of a quadrilateral are given in magnitude, the area is a maximum when the fourth side  $d$  is the diameter of the circle through the vertices; and generally,

When all the sides but one of a polygon of any order are given in magnitude, the area is a maximum when the circle on the closing side as diameter passes through the remaining vertices.†

[Proof as above.]

5. Having given of a quadrilateral the diagonals  $\delta$  and  $\delta'$  and a pair of opposite sides  $BC$  and  $AD$ , its area is a maximum when  $BC$  is parallel to  $AD$ .

[Take any position of the quadrilateral and through  $C$  draw  $CE$  parallel and equal to  $\delta$ . Join  $BE$  and  $AE$ .

The triangles  $BDE$  and  $BCD$  are equal (Euc. I. 37); to each add  $ABD$ , therefore  $ABCD = ABED$ .

\* The student should learn the proof of the Trigonometrical expression for the area of any quadrilateral in terms of the four sides and the sum of either pair of opposite angles.

$$(\text{Area})^2 = (s-a)(s-b)(s-c)(s-d) - abcd \cos^2 \frac{1}{2}(A+C).$$

(Casey's *Plane Trig.*, art. 152, cors. 3, 4.)

† To construct the quadrilateral. Let  $\theta$  be the angle between  $a$  and  $b$ , and  $AC=x$ .

$$\text{Then} \quad d^2 = c^2 + x^2 = a^2 + b^2 + c^2 - 2ab \cos \theta; \dots\dots\dots (1)$$

$$\text{but} \quad \cos \theta = -c/d;$$

substituting in (1) and simplifying we have the following expression for  $d$ :—

$$d^3 - d(a^2 + b^2 + c^2) - 2abc = 0,$$

an equation which has only one positive root. (Burnside and Panton's *Theory of Equations*, Art. 13.)

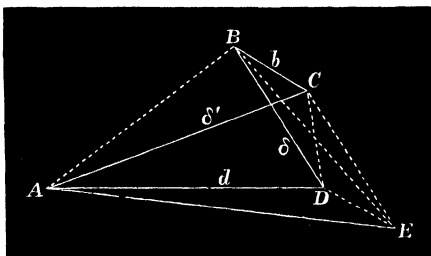
In the particular case when  $a = b = c$ , the equation for  $d$  reduces to

$$(d-2a)(d+a)^2 = 0;$$

hence

$$d = 2a,$$

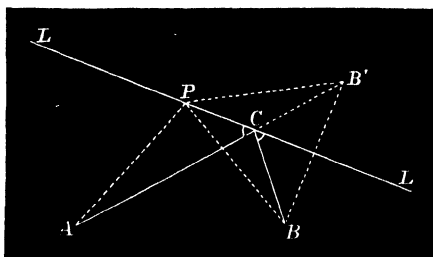
thus showing that the quadrilateral is half the regular inscribed hexagon.



Now,  $ABDE$  is a maximum when  $AD$  and  $DE$  are in the same straight line; hence  $ABCD$  is a maximum when  $BC$  is parallel to  $AD$ .]

6. The diagonals of a quadrilateral are 9 and 10 feet and two opposite sides 5 and 3 feet; find when its area is a maximum.

**10. Theorem.**—*Having given the base  $AB$  of a triangle and the locus of the vertex a line  $L$  meeting the base produced, the sum of the sides  $AC + BC$  is a minimum when  $L$  is the external bisector of the vertical angle.*



Let fall a perpendicular  $BL$  and make  $B'L = BL$ . Join  $AB'$  and let  $C$  be its intersection with  $L$ . Take any other point  $P$  on the line and join  $AP$  and  $B'P$ .

The triangles  $BCL$  and  $B'CL$  are equal in every respect (Euc. I. 4); hence  $BC = B'C$ . Similarly  $BP = B'P$ . Hence since (Euc. I. 20)  $AP + B'P > AB'$  it follows that  $AP + BP > AC + BC$ .

COR. 1. If the line  $L$  cuts the base internally the difference of the sides ( $AC - BC$ ) is a maximum when it bisects internally the angle  $C$ .

#### EXAMPLES.

1. The triangle of minimum perimeter inscribed in a given one is formed by joining the feet  $X, Y, Z$  of the perpendiculars \* let fall from the vertices on the opposite sides.

[For the joining lines are equally inclined to sides on which they intersect (Euc. III. 21).]

2. The polygon of least perimeter that can be inscribed in a given one is that whose angles are bisected externally by its sides. (By Ex. 1.)

3. The base and area of a triangle being given, the perimeter is least when the triangle is isosceles.

[For the line  $L$  is parallel to the base.]

4. If from  $O$ , the point of intersection of the diagonals of a cyclic quadrilateral, perpendiculars are drawn to the sides and their feet  $P, Q, R, S$  joined, the quadrilateral  $PQRS$  is of minimum perimeter.

4a. If points  $P', Q', R', S'$  be taken on the sides of the given quadrilateral, such that  $P'Q', Q'R', R'S'$  are parallel to  $PQ, QR, RS$ , then  $P'S'$  is parallel to  $PS$  and the perimeters of the quadrilaterals are equal. [Euc. VI. 2 and I. 5.]

5. The value of the minimum perimeter of the indeterminate inscribed quadrilateral in Ex. 4 is  $2\delta\delta/D$ , where  $D$  is the diameter of the circum-circle.

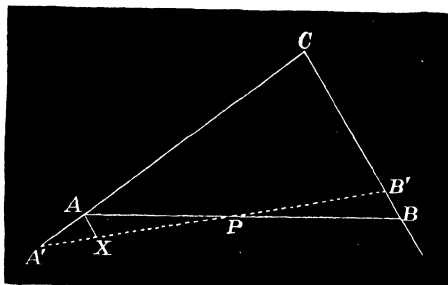
6. Given a triangle  $ABC$ , find a point  $O$  such that

$$OA + OB + OC \text{ is a minimum.}$$

[Where  $BOC = COA = AOB = 120^\circ$ .]

\* These are generally known as the *Perpendiculars of the Triangle*, and  $XYZ$  as the *Pedal Triangle* of  $ABC$ .

11. **Problem.**—Given an angle  $C$  of a triangle and a point  $P$  on the base, construct the triangle of minimum area.



Through  $P$  draw  $APB$  such that  $AP=BP$ . The triangle  $ABC$  is less than any other  $A'B'C$ .

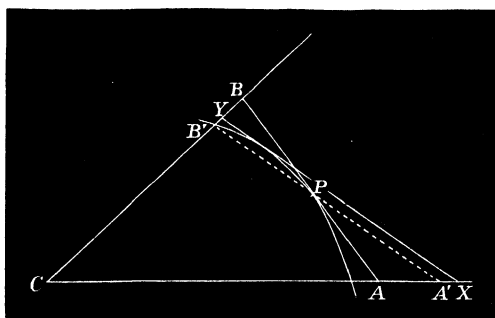
For draw  $AX$  parallel to  $BB'$ . Then the triangles  $APX$  and  $BPB'$  are equal in all respects (Euc. I. 4); hence  $AA'P > BB'P$ . To each add  $APB'C$ , therefore  $A'B'C > ABC$ ; hence the triangle of least area is that whose base is bisected at this point.

12. **Theorem.**—Given an angle and any curve concave to its vertex  $C$ . The tangent  $AB$  which forms with the sides of the angle a triangle  $ABC$  of minimum area is bisected at its point of contact ( $P$ ).

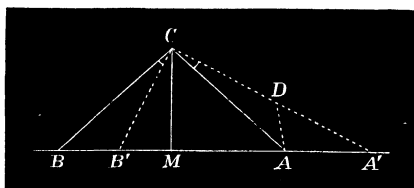
For this tangent cuts off a less area than any other line  $A'B'$  through  $P$ , because it is bisected at  $P$ . Now draw any other tangent  $XY$ , and let  $PA'B'$  be parallel to it. Since the curve is concave to  $C$ ,  $A'B'C < XYC$ ; a fortiori  $ABC < XYC$ .

COR. 1. In the particular case when the curve is a circle whose centre is at  $C$  the triangle is isosceles. This

property may be stated otherwise. *When the vertical angle and altitude of a triangle are given, the base and area are both minima when the triangle is isosceles.*



On account of its importance an independent proof of this property of the isosceles triangle is given.

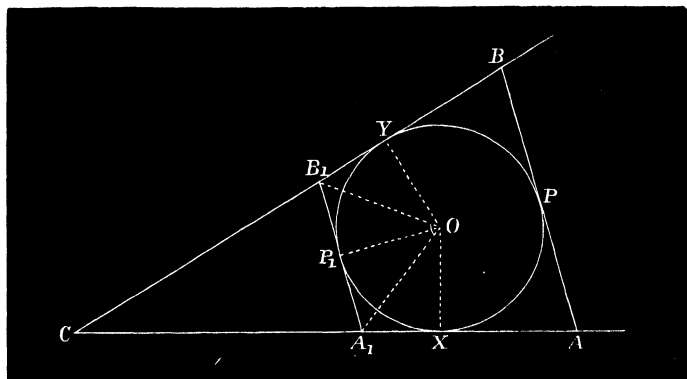


Let  $ABC$  be an isosceles triangle and  $A'B'C$  any other, having the same vertical angle and altitude  $CM$ .

Now  $BC > B'C$  (Euc. III. 8), but  $BC = AC < A'C$ , hence  $A'C > B'C$ . Let  $CD = B'C$ , join  $AD$ . The triangles  $ACD$  and  $BB'C$  are equal in every respect (Euc. I. 4), hence  $AA'C > BB'C$ ; therefore  $A'B'C > ABC$ ; therefore, etc.

**COR. 2.** When the curve is a circle touching the sides of the angle the tangent  $AB$  and area  $ABC$  are each minima when the triangle is isosceles.

COR. 3. If we consider the portion of the circle in Cor. 2, which is convex to  $C$ , the intercept of a variable tangent made by the sides of the angle subtends a constant angle  $\alpha$  at the centre of the circle ( $2\alpha = \pi - C$ ).



Hence the variable triangle  $A_1B_1O$  has a constant vertical angle ( $\alpha$ ) and altitude ( $\gamma$ ), and therefore its base and area are minima when  $A_1O = B_1O$ . In this case the point of contact  $P_1$  is the middle point of  $A_1B_1$ . Therefore, having given a circle and two fixed tangents, the portion of a variable tangent intercepted by the fixed tangents becomes a minimum in two positions, viz., when its point of contact bisects the arc  $XY$  internally or externally.

In the latter case the area cut off ( $ABC$ ) is a minimum but in the former a maximum ;

$$\text{For} \quad A_1B_1C = CXOY - 2A_1B_1O ;$$

therefore, since  $CXOY$  is constant, when  $A_1B_1O$  is a minimum,  $A_1B_1C$  is a maximum.



## EXAMPLES.

1. The triangle of least area and perimeter escribed to a circle is equilateral.

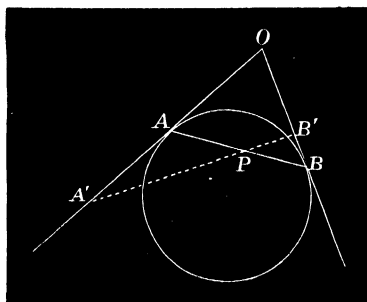
[For the point of contact of each side bisects the arc between the other; cf. Art. 8, Ex. 1.]

2. The polygon of least area and perimeter escribed to a circle is regular. (By Ex. 1.)

3. Having given the vertical angle  $C$  of a triangle in position and magnitude, and the in- or corresponding ex-circle, to prove that the line  $LM$  joining the middle points of the sides forms with the centre of the circle a triangle of constant area.

[For the ex-circle: if  $p$  be the perpendicular of  $ABC$  drawn from  $C$  to the base, and  $r_3$  the radius, we have  $2OLM = \frac{1}{2}c(\frac{1}{2}p + r_3) = \frac{1}{2}ABC + AOB = \frac{1}{2}OCXY = \text{const.}, \text{etc.}]$

**13. Problem.**—Given an angle  $O$  of a triangle and a point  $P$  on the base, construct it such that  $AP \cdot BP$  is a minimum.



Through  $P$  draw  $AB$  so that the triangle  $ABO$  is isosceles. Describe a circle touching the sides of the angle at  $A$  and  $B$ , and draw any other line  $A'PB'$ .

It is evident that  $AP \cdot PB < A'P \cdot B'P$ , and is therefore a minimum.

EXAMPLE.

1. Through the point of intersection  $P$  of two circles draw a line  $APB$  such that  $PA \cdot PB$  is a minimum.

[This reduces to describe a circle touching the two given ones at  $A$  and  $B$  such that  $A$ ,  $B$  and  $P$  are in a line.

It will be afterwards seen that this line passes through a point  $Q$ , on the line of centres  $O_1O_2$  of the circles where  $QO_1/QO_2$  = the ratio of the radii.]

**14. Theorem.**—*If a right line be divided into any two parts  $a$  and  $b$ , their rectangle is a maximum when the line is bisected.*

$$\text{For Euc. (II. 5)} \quad ab + \left(\frac{a-b}{2}\right)^2 = \left(\frac{a+b}{2}\right)^2 = \text{const.},$$

hence  $ab$  is a maximum when  $a-b=0$  or when  $a=b$ .

**COR.** The continued product of the segments of a line is a maximum when the parts are equal.

EXAMPLES.

1. Through any point  $P$  on the base of a triangle parallels  $PX$  and  $PY$  are drawn to the opposite sides; the area of the parallelogram  $PXCY$  is a maximum when the base  $AB$  is bisected at  $P$ .

[For the triangles  $APX$  and  $BPY$  are constant in species, hence  $PX \cdot PY \propto AP \cdot BP$ . But the area of the parallelogram  $= PX \cdot PY \sin C \propto PX \cdot PY$ ; therefore, etc.]\*

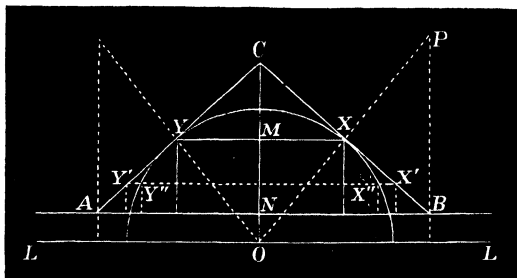
2. The maximum rectangle inscribed in a given segment of a circle is such that if tangents  $BC$  and  $AC$  be drawn at its vertices  $X$  and  $Y$ , then  $BX=CX$  and  $CF=AY$ .

[For  $NX$  is the maximum rectangle that can be inscribed in the triangle  $BCN$ , and therefore greater than any other  $X'N$ . Hence from the symmetry of the figure the rectangle on the side  $XY$  is greater than that on  $X'Y'$ , and a fortiori greater than that on  $X''Y''$ .

---

\* Hence, the maximum parallelogram inscribed in a triangle is half the area of the triangle.

The construction of the maximum rectangle is as follows :—Let  $BL$  be drawn perpendicular to  $OL$ , the diameter of the circle parallel to  $AB$ . Join  $OX$  and let it meet  $BL$  in  $P$ . Since the triangles  $OCX$  and  $BPX$  are equal in every respect (Euc. I. 26)



$PX = OX = r$ . Also  $OXBL$  is a cyclic quadrilateral, therefore Euc. (III. 36),

$$PB \cdot PL = PO \cdot PX = 2r^2,$$

but  $PL - PB$  is given ; hence the segment  $PB$  is known, and since it is equal to  $OC$ ,  $C$  is determined.

In the general case the line  $AB$  does not meet the circle, the segment is therefore imaginary, and the proposition may be thus stated :—given a line  $AB$  and a circle ; construct the maximum rectangle, having two of its vertices  $X$  and  $Y$  on the circle and the remaining two on the line.]

3. Draw a chord  $XY$  of a circle in a given direction such that the area of the quadrilateral  $ABXY$ , where  $AB$  is a given diameter, is a maximum.

[Draw a diameter  $YX'$ , and  $AYBX'$  is a rectangle, hence  $AX'$  is equal and parallel to  $BY$ . Join  $BX$  and  $XX'$ , and draw  $BC$  parallel to  $XX'$ .

Then since

$$\text{triangle } AX'X + \text{triangle } BXY = \text{triangle } ABX',$$

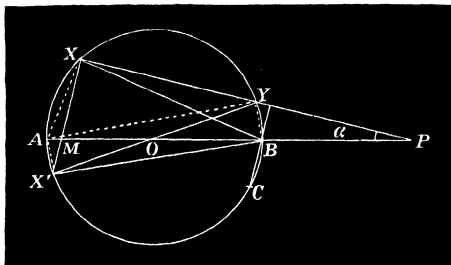
reject the common part  $AMX'$  and let  $BMX$  be added to each,

$$BXX' = ABXY.$$

The quadrilateral is a maximum therefore when  $BXX'$  is a

maximum. It is easy to see that the latter is half the rectangle inscribed in a *given* segment  $BC$ .

For since  $BC$  is parallel to  $XX'$ ,  $AC$  is perpendicular to  $XX'$  and therefore parallel to  $PX$ , hence  $BAC = \alpha$ .



The problem is thus reducible to Ex. 2.]

4. If a given finite line be divided into any number of parts  $a, b, c \dots$ ; to find when  $a^\alpha b^\beta c^\gamma \dots$  is a maximum, where  $\alpha, \beta, \gamma$  are given quantities.

[This expression is a maximum when

$$\left(\frac{\alpha}{a}\right)^\alpha \left(\frac{b}{\beta}\right)^\beta \left(\frac{c}{\gamma}\right)^\gamma \dots \text{ is a maximum} \dots \dots \dots (1)$$

but  $a/a$  is one of the  $\alpha$  equal parts into which the segment  $a$  may be divided; hence  $(a/a)^\alpha$  is the product of the equal subdivisions. Similarly  $(b/\beta)^\beta$  is the product of the  $\beta$  equal subdivisions of  $b$ , and so on. Therefore (1) attains its greatest value when the subdivisions of  $a, b, c \dots$  are all equal; *i.e.*, when

$$\frac{a}{\alpha} = \frac{b}{\beta} = \frac{c}{\gamma} = \dots]$$

5. Find a point  $O$  with respect to a triangle such that the product of the areas  $(BOC)(COA)(AOB)$  is a maximum.

[Since  $BOC + COA + AOB$  is constant, when  $BOC = COA = AOB$ , by Ex. 4, or when  $O$  is the centroid of the triangle.]

6. The maximum triangle of given perimeter is equilateral.

[From the formula  $\Delta^2 = s(s-a)(s-b)(s-c)$ ; since the sum of the factors on the right hand side is constant,  $\Delta$  is a maximum when  $s-a = s-b = s-c$ ; therefore, etc.]

7. The maximum parallelogram of given perimeter and angles is equilateral.

8. If  $p_1, p_2, p_3$  denote the perpendiculars from any point  $O$  on the sides of a triangle, the maximum value of  $p_1 p_2 p_3$  is  $8\Delta^3/27abc$ , and  $O$  is then the centroid of the triangle. (By Ex. 5.)

[Otherwise thus:—Since  $4abp_1 p_2 \equiv (ap_1 + bp_2)^2 - (ap_1 - bp_2)^2$  for any point  $O$  on the base  $c$ ,  $p_1 p_2$  is maximum when  $ap_1 - bp_2$  vanishes, since  $ap_1 + bp_2$  equals  $2\Delta$ . Then  $O$  is the middle point of the base. Now if  $p_3$  be supposed constant,  $O$  is on the median through  $C$ . Similarly by regarding  $p_1$  as constant,  $O$  would be found on the median through  $A$ ; and so on. Therefore if the three perpendiculars vary, their product is a maximum for the point of intersection of the medians.]

15. **Theorem.**—*If a right line be divided into any two parts  $a$  and  $b$  the sum of their squares is a minimum when the line is bisected.*

For (Euc. II. 9, 10)

$$a^2 + b^2 = 2\left(\frac{a+b}{2}\right)^2 + 2\left(\frac{a-b}{2}\right)^2.$$

Hence  $a^2 + b^2$  is minimum when  $a - b$  is minimum, because  $a + b$  is constant; that is when  $a = b$ .

COR. The sum of the squares of the segments of a line is a minimum when the segments are equal.

16. **Problem.**—*If a right line be divided into any number of parts  $a, b, c \dots$ , to find when*

$$\frac{a^2}{\alpha} + \frac{b^2}{\beta} + \frac{c^2}{\gamma} + \dots \text{ is minimum}$$

where  $\alpha, \beta, \gamma$  are known quantities.

Let the segment  $a$  be divided into  $\alpha$  equal parts; each part is therefore  $a/\alpha$  and the sum of squares of the parts  $= \alpha \left(\frac{a}{\alpha}\right)^2 = \frac{a^2}{\alpha}$ .

Similarly if the segment  $b$  be divided into  $\beta$  equal parts the sum of squares of the subdivisions  $= b^2/\beta$ ; and so on.

Hence the above expression denotes the sum of the squares of the subdivisions of the parts  $a, b, c \dots$ , and is therefore a minimum when these are equal; *i.e.*, when

$$\frac{a}{\alpha} = \frac{b}{\beta} = \frac{c}{\gamma} = \dots$$

EXAMPLES.

1. Divide a line into two parts  $a$  and  $b$  such that

$$3a^2 + 4b^2 \text{ is a minimum.}$$

[When  $\frac{a^2}{4} + \frac{b^2}{3}$  is minimum, *i.e.*, when  $\frac{a}{4} = \frac{b}{3}$ , hence  $3a = 4b$ .]

2. To find a point  $P$  such that the sum of the squares of its distances,  $x, y, z$ , from the sides of a triangle is a minimum.

[Let  $\Delta_1, \Delta_2, \Delta_3$  denote twice the areas of the triangles subtended by the sides of the given one at the point. Now since  $\Delta_1 = ax$ ,  $\Delta_2 = by$ , and  $\Delta_3 = cz$ ,

$$x^2 + y^2 + z^2 = \frac{\Delta_1^2}{a^2} + \frac{\Delta_2^2}{b^2} + \frac{\Delta_3^2}{c^2} \dots \dots \dots (1)$$

and is consequently a minimum when

$$\frac{\Delta_1}{a^2} = \frac{\Delta_2}{b^2} = \frac{\Delta_3}{c^2} \dots \dots \dots (2)$$

since  $\Delta_1 + \Delta_2 + \Delta_3 = \text{const.}$

From (2) it is obvious that  $\frac{x}{a} = \frac{y}{b} = \frac{z}{c} \dots \dots \dots (3)$

This result may also be seen from the identity

$$(a^2 + b^2 + c^2)(x^2 + y^2 + z^2) - (ax + by + cz)^2 \\ \equiv (bz - cy)^2 + (cx - az)^2 + (ay - bx)^2,$$

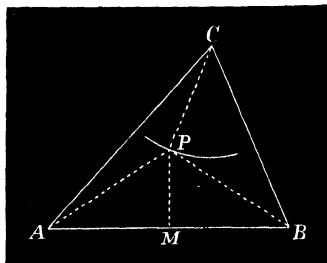
with which the student should be familiar.]

NOTE.—This point is termed the *Symmedian Point* of the triangle, as it is obvious from (3) that the lines joining it to the vertices of the given triangle make the same angles with the sides as the corresponding medians; also since

$$\frac{x}{a} = \frac{y}{b} = \frac{z}{c} = \frac{ax + by + cz}{a^2 + b^2 + c^2} = \frac{2\Delta}{a^2 + b^2 + c^2}, \\ x = \frac{2a\Delta}{a^2 + b^2 + c^2}, y = \frac{2b\Delta}{a^2 + b^2 + c^2}, z = \frac{2c\Delta}{a^2 + b^2 + c^2}.$$

3. Find a point  $P$  such that the sum of squares of its distances from the vertices of a triangle may be a minimum.

[If  $CP$  be supposed constant while  $AP$  and  $BP$  vary, the point  $P$  describes a circle around  $C$  as centre, and if  $M$  be the middle point of the base  $AP^2 + BP^2 = 2AM^2 + 2MP^2$ . Hence  $AP^2 + BP^2 + CP^2$  is minimum when  $2PM^2 + CP^2$  is minimum, since  $AM$  is constant. Therefore  $P$  is a point on the median  $CM$  such that  $CP/PM = 2$ , i.e., the centroid.



Similarly by supposing  $AP$  or  $BP$  to remain constant we find the same point. Hence the centroid is the required point when  $AP$ ,  $BP$  and  $CP$  all vary.]

## SECTION II.

### METHOD OF INFINITESIMALS.

17. It has probably been observed in the preceding section that the positions of maximum and minimum of a quantity, varying according to a given law, are symmetrical with respect to the fixed parts of the figure. Thus when the base and vertical angle of a triangle are given, the altitude, rectangle under sides, area, etc., etc., are maxima when the triangle is *isosceles*.

In Art. 9 the triangle of maximum area is found by placing the two given sides at *right* angles:

Again, a figure of given perimeter and of maximum area is *circular*. As the variable line  $AB$  in Art. 11 rotates in a positive direction around  $P$ , according as  $PB$  recedes from the perpendicular from  $P$  on  $BC$ , the segments  $AP$  and  $BP$  approach an equality, and the triangle  $ABC$  is a minimum when  $AP=BP$ .

18. The several parts, of a geometrical figure which varies according to a definite law, can always be expressed in terms of the fixed parts of the figure and those quantities which are sufficient to define its position.

Take for example the figure of Art. 8. In any position of the vertex  $C$ , by assuming the triangle to be of given altitude; the variable parts,  $a$ ,  $b$ , area, and other functions of the sides or angles can be found in terms of the base  $c$ , vertical angle  $C$ , and altitude.

Thus the variables may be regarded as functions of the given parts and the *co-ordinates* of their position.

It follows, then, that if the latter vary continuously those functions must do likewise.\* Hence a very small change in position will cause a very slight change or *increment* in the magnitude of the function. Suppose in Art. 8 the circle to be divided into an indefinitely great number of equal parts, and let the vertex  $C$  occupy each point of section from  $A$  towards  $B$ . As the *altitude* thus receives indefinitely small increments so does the area.

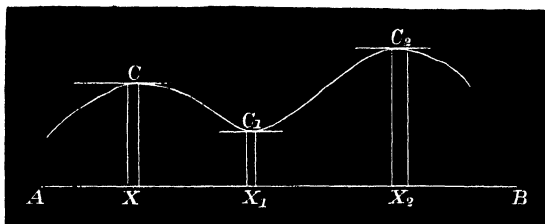
Let  $AB$  be the base of a triangle and any curve  $CC_1C_2$  the locus of its vertex.

---

\* See Burnside and Panton's *Theory of Equations*, Art. 7.



In the figure as the vertex approaches  $C$  on the curve from left to right the intercept  $AX$  made by the perpendicular may be taken as the co-ordinate of its position, since if  $AX$  is known the position of  $C$  is also known.



Thus while  $AX$  continues to receive positive increments, the area, altitude, and other functions of it are sometimes decreasing, as from  $C$  to  $C_1$ , and sometimes increasing, as from  $C_1$  to  $C_2$ .

At the points  $C, C_1, C_2$  the increments in the altitude alter in sign *and therefore consecutive values are equal*. Here also the tangents to the curve are parallel to the base  $AB$ , and at any other point  $C_n$  the increment of the variable divided by the corresponding increment in the function  $= \cot \alpha$ , where  $\alpha$  is the angle made by the tangent at  $C_n$  with  $AB$ . We have seen that if  $AX$  denote the value of a variable in any position, and  $CX$  any function of  $AX$ , when the function passes through a maximum or minimum its two consecutive values are in each case equal to one another.

Suppose, for example, that a variable chord  $XY$  of a circle moves parallel to a certain direction; it gradually increases in length as it approaches the centre and if  $XY$

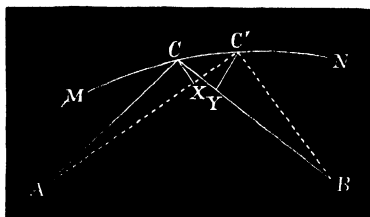
be a diameter and  $X'Y'$  a consecutive chord; since  $XX'$  and  $YY'$  are tangents to the circle, and therefore parallel,  $XYX'Y'$  is a parallelogram and  $XY = X'Y'$  (Euc. I. 34). Hence the diameter is the maximum chord in a circle (cf. Euc. III. 15).

# EXAMPLES.

1. Having given the base and locus of vertex of a triangle; find when the area is a maximum or minimum.

[Let the locus be a curve of any order and it is readily seen (Euc. I. 39) that the tangents at the required points are parallel to the base.]

2. In Ex. 1 when is the sum of the sides a minimum or maximum?



[Let  $C$  and  $C'$  be two points indefinitely near to each other on the locus  $MN$ . Draw  $CX$  and  $C'Y$  perpendiculars to  $AC'$  and  $BC$  respectively.

Then since in the triangle  $ACX$ ,  $X$  is a right angle and  $A$  indefinitely small,  $ACX$  is approximately a right angle and  $AC$  is nearly equal to  $AX$ . Hence in the limit

$$C'X = AC' - AX = AC' - AC.$$

Similarly  $CY$  is the increment (negative) of  $BC$ .

Therefore  $C'X = CY$  and the right-angled triangles  $CC'X$  and  $CC'Y$  are equal in every respect, and  $\angle AC'C = \angle BCC'$ . But  $AC'C = ACM$  when  $A$  is indefinitely small; hence the required points  $C$  on the locus are such that  $AC$  and  $BC$  are equally inclined to the curve, i.e., to the tangent at their point of intersection.

It similarly follows that if  $A$  and  $B$  were upon opposite sides of the curve this relation holds when  $AC - BC$  is maximum or minimum.\*]

3. Given the vertex  $A$  of a triangle fixed, the angle  $A$  in magnitude and the base angles moving on fixed lines intersecting in  $O$ ; to construct the triangle  $ABC$  of minimum area.

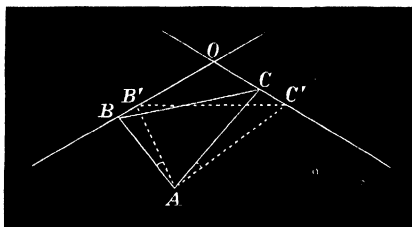
[By taking two consecutive positions as in figure, we have

$$AB \cdot AC = AB' \cdot AC' \text{ and } \angle BAB' = \angle CAC'.$$

Hence

$$AB : AB' = AC' : AC,$$

and the triangles  $BAB'$  and  $CAC'$  are similar (Euc. VI. 6).]



Therefore  $\angle ABO = \angle AC'O = \angle ACO$  in the limit

In the required position the sides  $AB$  and  $AC$  are *equally* inclined to the given lines. Here again we have an illustration of the symmetry of the figure when the triangle is minimum. If the angle  $A$  is  $180^\circ$  the property (Art. 13) follows at once.]

4. Given two sides of a triangle fixed in position and a point  $P$  on the base; when is  $AB$  a minimum?

[Taking two consecutive positions of  $AB$  and drawing perpendiculars  $AX$  and  $BY$ ; as before  $A'X$  is the increment of  $AP$  and  $B'Y$  of  $BP$ ; hence  $A'X = B'Y$ .

$$\text{Again} \quad A'X = AX \cot A' = AP \sin P \cdot \cot A'.$$

$$\text{Similarly} \quad B'Y = BY \cot B = BP \sin P \cot B.$$

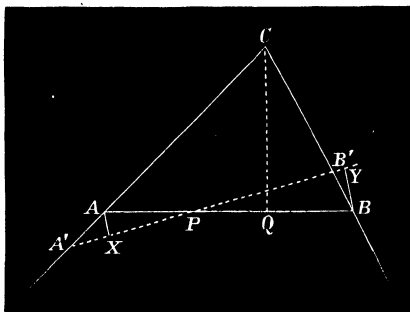
Therefore in the limit

$$AP \cot A = BP \cot B.$$

---

\* It follows if the curve is of such a nature that  $AC + BC$  is constant then for every point on it  $AC$  and  $BC$  are equally inclined.

But if  $Q$  denote the foot of the perpendicular on the base we have



$$BQ \cot A = AQ \cot B,$$

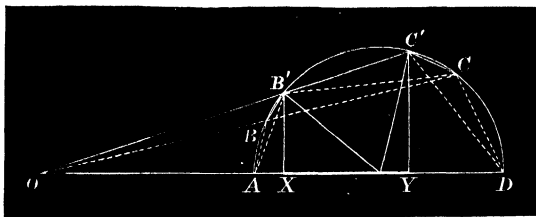
hence

$$AP=BQ,$$

or the minimum chord is such that the *given point P* and the foot of the perpendicular are equidistant from the extremities of the base.

This is known as Philo's Line.

5. Through a given point  $O$  in the diameter produced of a semi-circle to draw a secant  $OBC$  such that the quadrilateral  $ABCD$  may be a maximum.



[Take two consecutive positions of the secant  $OBC$  and  $OB'C'$  such that  $ABCD = AB'C'D$ , and join  $AB$ ,  $AB'$ ,  $DC$ ,  $DC'$ , and  $B'C$ .

Now since  $ABCD = AB'C'D$  it follows that

$$BB'CC' = ABB' + DCC'$$

**or**

$$BB'C + CB'C' = BB'A + CC'D.$$

Transposing we have

$$BB'C - BB'A = CC'D - CB'C',$$

or since twice the area of a triangle is the product of two sides  $\times$  the sine of the included angle ; in the limit this relation becomes

$$\frac{BB'(BC^2 - AB^2)}{\text{diameter}} = \frac{CC'(CD^2 - BC^2)}{\text{diameter}} ;$$

but from similar triangles  $BB'/CC' = OB/OC$ . Hence if  $AB = a$ ,  $BC = b$ ,  $CD = c$ ,  $AD = d$ , and the angles subtended at the centre of the circle by the sides  $a, b, c$  be denoted by  $2\alpha, 2\beta, 2\gamma$ , this relation may

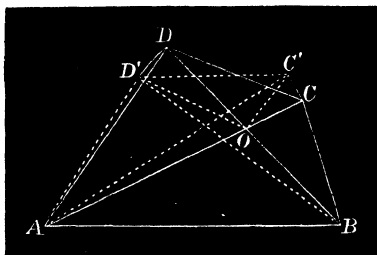
be written 
$$\frac{b^2 - a^2}{c^2 - b^2} = \frac{OC}{OB},$$

which is easily reducible to

$$\cos 2\alpha + \cos 2\gamma = 1,$$

or the *projection XY of the intercept is equal to the radius of the circle*. The construction of the chord  $BC$  will be afterwards given.]

6. Having given two opposite sides  $AB$  and  $CD$  of a quadrilateral and the diagonals  $CA$  and  $BD$ , to construct it so that the area may be a maximum.



[Let  $AB$  be fixed and draw  $C'$  and  $D'$  consecutive positions of  $C$  and  $D$ . Let  $O$  be the intersection of  $AC$  and  $BD$ . Then since  $CC'$  is small compared with  $OC$  and  $OC C'$  a right angle ;  $OC C'$  may be considered an isosceles triangle, and  $OC = OC'$ . Similarly  $OD = OD'$  ; and since  $CD = C'D'$  the triangles  $COD$  and  $C'OD'$  are equal in every respect. From the equal areas  $ABCD$  and  $ABC'D'$  take the equals  $COD$  and  $C'OD'$  and the common part  $AOB$ , and there remains  $BOC + AOD = BOC' + AOD'$ ,

or

$$BOC' - BOC = AOD - AOD',$$

hence

$$BO \cdot CO = AO \cdot DO,$$

from which it is manifest that  $CD$  and  $AB$  are parallel. Cf. Art. 9, Ex. 5.

A similar proof may be applied to show that when the four sides of a quadrilateral are given the area is a maximum when

$$CO \cdot AO = BO \cdot DO,$$

i.e., when the figure is cyclic. See Milne's *Companion to the Weekly Problem Papers*, 1888, p. 27.]

7. To draw a parallel to a given line meeting a semicircle in  $C$  and  $D$  such that  $ABCD$  is a quadrilateral of maximum area.

[As before, when  $ABCD$  is a maximum it is equal to the consecutive area  $ABC'D'$ .

Hence

$$CC'DD' = \triangle CC' + BDD',$$

therefore

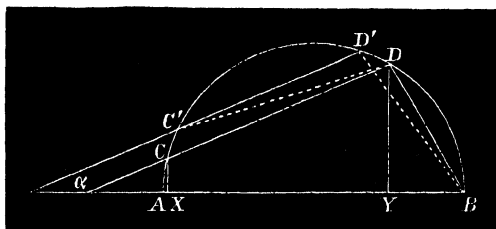
$$CC'D - CC'A = DD'B - DD'C',$$

which in the limit reduces to

$$b^2 - a^2 = c^2 - b^2 \text{ or } 2b^2 = a^2 + c^2 \dots \dots \dots (1)$$

Again if  $X$  and  $Y$  are the projections of  $C$  and  $D$  on the diameter  $d$  of  $AB$  we have

$$AX = a^2/d, BY = c^2/d \text{ and } XY = b \cos \alpha.$$



Making these substitutions in (1) we have on reducing

$$2b^2 + d \cos \alpha \cdot b - d^2 = 0 \dots \dots \dots (2)$$

NOTE.—If  $\alpha = 0$  the quadrilateral is found to be one half of the inscribed hexagon.

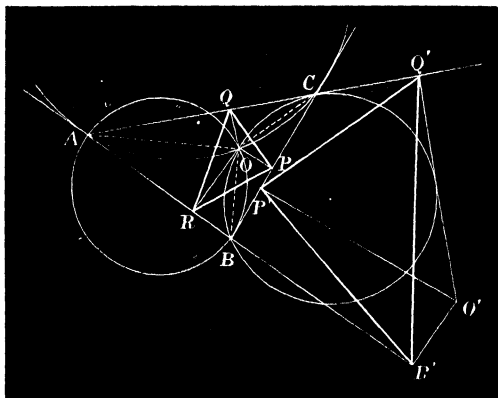
If  $\alpha = 90$  the maximum quadrilateral is the inscribed square.

## SECTION III.

## THE POINT O THEOREM.

**19. Theorem.**—*If points  $P$ ,  $Q$ , and  $R$  be taken on the sides of a triangle the circles  $AQR$ ,  $BRP$ , and  $CPQ$  pass through a common point  $O$ .*

For let the circles  $AQR$  and  $BRP$  meet in  $O$ . Then since (Euc. III. 22)  $QOR = \pi - A$  and  $ROP = \pi - B$ , we have  $QOP = 2\pi - (\pi - A) - (\pi - B) = A + B = \pi - C$ ; therefore the quadrilateral  $POQC$  is cyclic.



*The angles  $BOC$ ,  $COA$ ,  $AOB$ , subtended by the sides of the given triangle at  $O$ , are respectively  $A + P$ ,  $B + Q$ ,  $C + R$ , when  $O$  is within the triangle  $ABC$ .*

For, applying Euc. I. 32 to the triangles  $BOC$  and  $COA$ , it follows that  $\angle AOB = C + CAO + CBO$ .

But  $CAO = QRO$  since  $AQRO$  is cyclic,  
 also  $CBO = PRO$  since  $BPRO$  is cyclic;  
 therefore  $AOB = C + R$ ..... (α)  
 where  $R$  denotes an angle of the triangle  $PQR$ . Similarly  
 for the angles  $BOC$  and  $COA$ .

If  $O$  falls outside the triangle  $ABC$  these angular relations become somewhat modified. Take for example  $O$  within the angle  $C$ .

Then from the cyclic quadrilaterals  $QRAO$  and  $RPBO$  we have (Euc. III. 20)

$$\angle ORP = OBP \text{ and } \angle ORQ = OAQ;$$

adding these equations

$$R = OAQ + OBP = C + AOB,$$

or  $AOB = R - C.$

Again, since Euc. I. 32,

$$A + ACO = BOC + ABO,$$

by transposing

$$A - BOC = ABO - ACO \dots\dots\dots (1)$$

But  $ABO = RPO$  since  $PRBO$  is cyclic,

and  $ACO = QPO$  since  $PQCO$  is cyclic.

Substituting these values in (1) we have

$$A - BOC = RPO - QPO = P;$$

therefore  $BOC = A - P.$

Similarly  $COA = B - Q$ ..... (β)

It may be shown in the same manner that if the points  $P, Q, R$  are such that two of the angles  $P, Q$  of the triangle formed by them are greater than  $A$  and  $B$  respectively

$$BOC = P - A,$$

$$COA = Q - B, \dots\dots\dots (\gamma)$$

and  $AOB = C - R.$

Hence if a triangle  $PQR$  of given species be inscribed in a given one  $ABC$ , the circles  $AQR$ ,  $BRP$ , and  $CPQ$



pass through either of two fixed points, one of which subtends at the sides of  $ABC$ , angles  $A + P$ ,  $B + Q$ ,  $C + R$ , and the other  $A - P$ ,  $B - Q$ ,  $R - C$ , or  $P - A$ ,  $Q - B$ ,  $C - R$ , according as two of the angles of the given triangle are greater or less than the corresponding angles of the inscribed triangle.

20. Let  $PQR$  be a triangle of given species inscribed in  $ABC$ . We have seen that the point  $O$  is fixed, and therefore the lines  $AO$ ,  $BO$  divide the angles of  $ABC$  into known segments. But the segments of  $A$  are equal to the base angles of the triangle  $QOR$ ; similarly of  $B$  to the base angles of  $ROP$ , and of  $C$  to the base angles of  $POQ$ .

Hence each of the triangles  $POQ$ ,  $QOR$ ,  $ROP$  are given in species. Therefore as the inscribed triangle  $PQR$  varies in position  $OQR$ ,  $ORP$ ,  $OPQ$  remain constant in species, and  $OP : OQ : OR$  are constant ratios.

Again, since the triangle  $OPQ$  is fixed in species and one vertex  $O$  a fixed point; if  $P$  describes a line  $BC$  it follows that the locus of  $Q$  is also a line ( $CA$ ). And generally, *when one vertex of a figure of given species is fixed and any other vertex  $P$  or point invariably connected with it describe a locus, the remaining points  $Q \dots$  describe loci, which may be derived from  $P$  by revolving it through a known angle  $POQ$  and increasing or diminishing  $OP$  in the ratio of  $OQ : OP$ .*

The loci thus described are similar, the ratio  $OP : OQ$  is termed their *Ratio of Similitude* and the point  $O$  the *Centre of Similitude*.

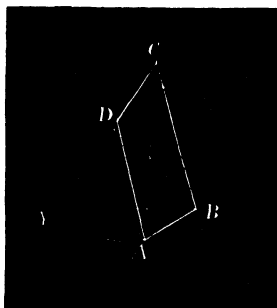
Thus since  $O$  is a point invariably connected with a variable inscribed triangle  $PQR$  of given species, the

ortho-centre, circum-centre, ex-centres, median point, etc., and all other points invariably connected with the triangle, describe right lines which can at once be constructed by the above method.

Moreover, we know that if  $O$  is fixed and  $P$  describes a circle, and the variable line or *Radius Vector*  $OP$  be divided in  $Q$ , in a given ratio, the locus of  $Q$  is a circle. Now if  $Q$  be turned around  $O$  through any given angle the locus is the same circle displaced through the same angle. Therefore if *one vertex of a triangle of given species is fixed, and another vertex describe a circle, the remaining vertex and all other points invariably connected with it likewise describe circles.*

#### EXAMPLES.

1. Having given the diagonals and angles of a quadrilateral  $ABCD$ , construct it.



[On one diagonal  $AC$  describe segments of circles containing angles respectively equal to  $B$  and  $D$ . Let  $ABCD$  be the required quadrilateral. Produce  $CD$  to  $Y$  and  $BC$  to  $X$ . Join  $BY$  and  $AY$ .

Then since the chord  $BY$  of a given circle subtends a given angle  $C$  it is of known length. The triangle  $ADY$  is also given in species; hence the following construction:—On  $BY$  describe a segment of a circle containing an angle  $C$ . The triangle  $ADY$ , of given species,

has one vertex  $Y$  fixed, another  $A$  describing the circle  $AFC$ , therefore the remaining vertex  $D$  describes a circle. Take  $B$  as centre and  $BD$  as radius, and cut this locus in the point  $D$ ; therefore, etc.\*]

2. Required to place a parallelogram of given sides with its vertices on four concurrent lines (M'Vicker).

[Let  $ABCD$  be the parallelogram situated on the pencil  $O$ .  $ABCD$ . Through  $C$  and  $D$  draw parallels  $CP$  and  $DP$  to  $BO$  and  $AO$  respectively. Join  $OP$ . By Ex. 1 the diagonals and angles of the quadrilateral  $CDPO$  are given; therefore, etc.]

21. When the triangle  $PQR$  is given in every respect, the triangles  $OPQ$ ,  $OQR$ ,  $ORP$  are completely determined; for in addition to their species we are given the sides  $PQ$ ,  $QR$ , and  $RP$ , hence the sides  $OP$ ,  $OQ$ ,  $OR$  are easily determined. We have therefore four solutions, real or imaginary, to the problem:—

*Having given two triangles  $ABC$  and  $PQR$  to place either with its vertices on the corresponding sides of the other; for having determined the point  $O$ , the position of which depends altogether on the species of the triangles, we get the position of the vertex  $P$  by taking  $O$  as centre and  $OP$  as radius and describing a circle cutting  $BC$ .*

22. When the line  $OP$  is perpendicular to  $BC$ ,  $OQ$  and  $OR$  are therefore perpendiculars to  $CA$  and  $AB$  respectively, and the circle with  $O$  as centre and  $OP$  as radius touches  $BC$ . In this case the two solutions coincide, and  $PQR$  is the minimum triangle of given species that can be inscribed in  $ABC$ .

23. It is manifest that a given triangle  $ABC$  may be escribed to another  $PQR$ . For having determined the point  $O$ , the triangles  $BOC$ ,  $COA$ , and  $AOB$  are given in species, and are therefore completely determined, since

---

\* For other solutions see "*Mathematics from the Educational Times*," Vol. XLIV., p. 29, by D. Biddle and Rev. T. C. Simmons.

$BC$ ,  $CA$ , and  $AB$  are given lines. Hence any vertex ( $C$ ) is found by describing a segment of a circle upon  $PQ$  containing an angle equal to  $C$ , and with  $O$  as centre and  $OC$  as radius describing circle. Where these circles meet is the required position of  $C$ .

Again in the triangle  $BOC$  when  $BC$  is a maximum  $OC$  is a maximum, and is therefore a diameter of the circle  $OPQC$ . Then  $OPC$  is a right angle. Hence *the maximum triangle of given species escribed to a given one is that whose sides are perpendicular to  $OP$ ,  $OQ$ ,  $OR$ .*

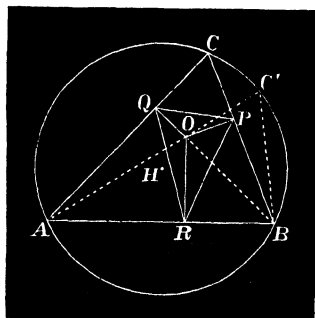
COR. If the sides of the given escribed triangle be  $\lambda$ ,  $\mu$ , and  $\nu$ , and  $\alpha$ ,  $\beta$ ,  $\gamma$  the distances of  $O$  from  $P$ ,  $Q$ ,  $R$ ,

$$\lambda\alpha + \mu\beta + \nu\gamma = \text{a minimum.}$$

Hence *required to find a point, given multiples of whose distances from three fixed points is a minimum when any two of the multiples are together greater than the third.*

#### EXAMPLES.

1. If  $d$  denote the distance of the point  $O$  from the circumcentre  $H$  of the triangle  $ABC$ ; prove that twice the area of the minimum triangle  $PQR$  is  $(R^2 \sim d^2) \sin A \sin B \sin C$ .



[For join  $AO$  and produce it to meet the circum-circle again in  $C'$ ; join  $BC'$ .

Now since  $\angle R = AOB - C = AOB - C' = OBC'$  (Euc. I. 32),  
 we have  $2PQR = RP \cdot RQ \sin R = RP \cdot RQ \sin OBC' \dots\dots(1)$   
 but  $RP = OB \sin B$  and  $RQ = OA \sin A$ .

Substituting these values in (1) and putting

$$\begin{aligned} OB \sin OBC' &= OC' \sin C', \\ 2PQR &= AO \cdot BO \sin A \sin B \sin OBC' \\ &= AO \cdot OC' \sin A \sin B \sin C \\ &= (R^2 \sim d^2) \sin A \sin B \sin C. \end{aligned}$$

NOTE.—If the point  $O$  is on the circum-circle  $R=d$  and the area of the triangle vanishes, hence if from any point on the circum-circle of a triangle perpendiculars be let fall upon the sides their feet lie in a line. This is termed a *Simson Line* of the triangle, and the collinearity of the points admits of an easy direct proof.

2. If the pedal triangle  $PQR$  of a point  $O$  is constant in area the locus of the point is a circle.

[Concentric with the circum-circle by the equation of Ex. 1.]

2a. The theorem holds generally for a polygon.

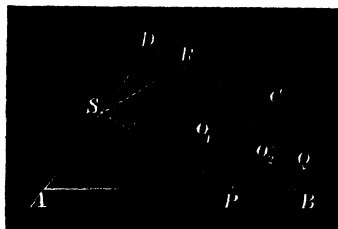
3. Having given of a triangle the base  $c$ , and  $ab \sin (C - \alpha)$  where  $\alpha$  is a given angle, find the locus of the vertex.

[In Ex. 1 we have

$$\begin{aligned} 2PQR &= AO \cdot BO \sin A \sin B \sin (AOB - C) \\ &\propto AO \cdot BO \sin (AOB - C), \end{aligned}$$

and the locus of  $O$  is in that case a circle. Hence in the triangle  $AOB$  we have the data in question; therefore the locus of the vertex is a circle concentric with  $H$ .]

4. To inscribe a quadrilateral of given species  $PQRS$  in a given quadrilateral  $ABCD$ .



Find the point  $O_1$  of the triangle  $PQR$  of given species inscribed in a given one, viz., that formed by three of the sides,  $AB$ ,  $BC$ ,  $CD$  of

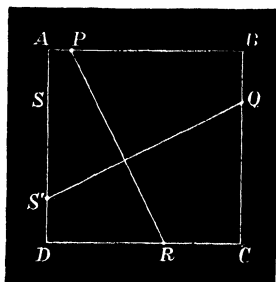
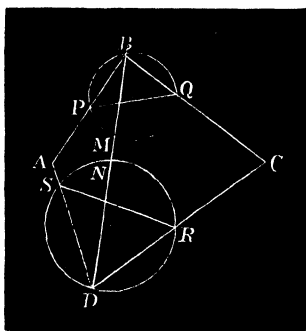
the quadrilateral. Similarly find  $O_2$  of the triangle  $PQS$  inscribed in a given one. Now by Art. 19, since the species of each of the triangles  $O_1PQ$  and  $O_2PQ$  is given, we have  $\angle O_1PO_2 = O_2PQ \sim O_1PQ =$  a known quantity ; therefore the point  $P$  is determined.

5. To escribe a quadrilateral  $ABCD$  of given species to a given one  $PQRS$ .

[Take any quadrilateral  $abcd$  of the same species as  $ABCD$ . Inscribe in it by Ex. 4 a quadrilateral  $pqrs$  of the species  $PQRS$ . It is obvious that  $\angle SPA = spa$ , since the figures are similar, hence the problem reduces to drawing lines in known directions through  $P, Q, R, S$ .

Otherwise thus :—

Upon a pair of opposite sides  $PQ$  and  $RS$  describe segments of circles containing angles equal to  $B$  and  $D$  respectively. Find a point  $M$  such that the arcs  $PM$  and  $QM$  subtend angles equal to  $ABD$  and  $CBD$  respectively. Similarly find  $N$  such that  $CDN$  and  $ADN$  may be equal to the known segments of the angle  $C$ . Join  $MN$  ; where it meets the circles in  $B$  and  $D$  are two of the required vertices of the quadrilateral  $ABCD$ .]

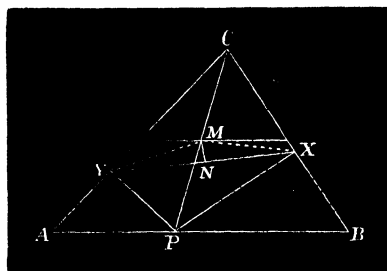


6. To escribe a square  $ABCD$  to a quadrilateral  $PQRS$ .

[By Ex. 5 or simply thus :—Join  $PR$  and let fall a perpendicular from  $Q$  upon it. Make  $QS' = PR$ .  $SS'$  is a side of the required square. This construction depends upon the property that *any two rectangular lines terminated by the opposite sides of a square are equal to one another* (Mathesis).

7. From any point  $P$  on the base of a triangle perpendiculars  $PX$  and  $PY$  are drawn to the sides, find the locus of the middle point  $N$  of  $XY$ .

[Bisect  $CP$  in  $M$ , join  $MX$ ,  $MY$  and  $MN$ . It is easy to see that  $MXY$  is an isosceles triangle of given species, each of its base angles being the complement of  $C$ ; and since its vertices  $X$ ,  $M$ ,  $Y$  move on fixed lines, any point  $N$  invariably connected with it describes a line. By taking  $P$  to coincide alternately with  $A$  and  $B$  the locus is seen to be the line joining the middle points of the perpendiculars from the extremities of the base of the triangle  $ABC$ .]



8. The sides of the pedal triangle  $PQR$  are in the ratios

$$a : AO : b : BO : c : CO.$$

[For  $QR = AO \sin A \propto a \cdot AO$ , etc.]

9. **Extension of Ptolemy's Theorem.**—If the three pairs of opposite connectors of four points be denoted by  $a, c$ ;  $b, d$ ;  $\delta, \delta'$  to prove the relation

$$\delta^2 \delta'^2 = a^2 c^2 + b^2 d^2 - 2abcd \cos (\theta + \theta'),$$

where  $\theta + \theta'$  is the sum of a pair of opposite angles of the quadrilateral.

[Let  $A, B, C, O$  be the four points. From any one of them  $O$  let fall perpendiculars  $OP, OQ, OR$  on the sides of the triangle  $ABC$  formed by the remaining three; then since

$$PQ^2 = QR^2 + RP^2 - 2QR \cdot RP \cos R,$$

substituting for  $PQ, QR, RP$  the values in Ex. 8, and reducing, the above equation follows at once (M'Cay).]

9a. What does this theorem reduce to for the quadrilateral  $ABCP$  in the figure of Ex. 7? Deduce the relation of Art. 3, Ex. 5, as a further particular case.

10. A variable circle passes through the vertex of an angle and a second fixed point; find the locus of the intersection of tangents at the extremities of its chord of intersection.

11. If  $\alpha, \beta, \gamma$  denote the distances of any point  $O$  from the sides of a triangle; to prove that

$$\alpha\beta\gamma = \frac{SS'}{2R}$$

where  $S$  and  $S'$  are the rectangles under the segments of a variable chord through  $O^*$  of the circum-circles of  $ABC$  and of the pedal triangle of the point  $O$  (M'Vicker).

[In Ex. 1 let  $K$  be the point where  $RO$  meets the circum-circle of  $PQR$ ; then  $\gamma = S'/OK = S' \sin P / \beta \sin OK$ .

But  $\sin OK = \sin(A + P) = \sin BOC$ ;  $\therefore \beta\gamma = S' \sin P / \sin BOC$ . Also  $\alpha = OB \cdot OC \sin BOC / a$ , therefore  $\alpha\beta\gamma = S' \cdot OB \cdot OC \sin P / a$ .

Again  $OB = RP / \sin B$ , etc. ... therefore by substitution

$$\alpha\beta\gamma = \frac{S' \cdot RP \cdot PQ \sin P}{a \sin B \sin C} = \frac{S' \cdot PQR}{\Delta \cdot 2R} = \frac{SS'}{2R}]$$

12. In the particular cases when  $O$  coincides with the in- or ex-centres of the triangle  $ABC$ , the formula in Ex. 11 reduces to

$$\delta^2 = R^2 - 2Rr \text{ or } \delta_1^2 = R^2 + 2Rr, \text{ etc.}$$

**24. Theorem.**—When three points  $P, Q, R$  are taken collinearly on the sides of a triangle, the circles circumscribing the four triangles  $QRA, RPB, PQC, ABC$  meet in a point.

This theorem may be easily proved directly, but it is obviously a particular case of Art. 19, for the circles  $QRA, RPB, PQC$  meet in a point  $O$  (Art. 19) such that  $COA = Q - B$ , which in this case is  $180 - B$ ; therefore, etc. Euc. III. 22.

The transversal  $PQR$  to the sides of  $ABC$  is the limiting case of a triangle inscribed in  $ABC$ , the angles at  $P$  and

---

\*The constant rectangle under the segments of a variable chord of a circle passing through a fixed point has been termed by Steiner the *Power of the Point* with respect to the circle.





2. Draw a line across a quadrilateral, meeting the sides in  $PQRS$  such that the ratios  $PQ : QR : RS$  may be given.

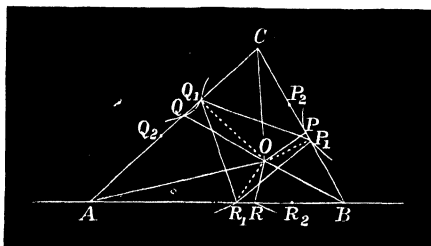
3. The line joining  $O$  to the orthocentre of  $ABC$  is bisected by the Simson line  $PQR$ , and intersects it on the nine points circle.

4. The angle subtended by any two points  $O_1$  and  $O_2$  on the circle is equal to the angle between their Simson lines.

5. The Simson lines of two points diametrically opposite intersect at right angles on the nine points circle. (By Ex. 4.)

**25. Theorem.** — For three positions,  $PQR$ ,  $P_1Q_1R_1$ ,  $P_2Q_2R_2$ , of the triangle of given species inscribed in a given one  $ABC$ ; to prove that

$$PP_1 : PP_2 = QQ_1 : QQ_2 = RR_1 : RR_2.$$



Since the triangles  $OPQ$ ,  $OP_1Q_1$ ,  $OP_2Q_2$  are similar, we have  $OP : OP_1 = OQ : OQ_1$ , also  $\angle POP_1 = \angle QQ_1$  since  $\angle POQ = \angle P_1OQ_1$ ; therefore the triangles  $POP_1$  and  $QQOQ_1$  are similar. Hence

$$PP_1 : QQ_1 = OP : OQ.$$

Similarly

$$QQ_1 : RR_1 = OQ : OR;$$

therefore

$$PP_1 : QQ_1 : RR_1 = OP : OQ : OR.$$

Similarly

$$PP_2 : QQ_2 : RR_2 = OP : OQ : OR;$$

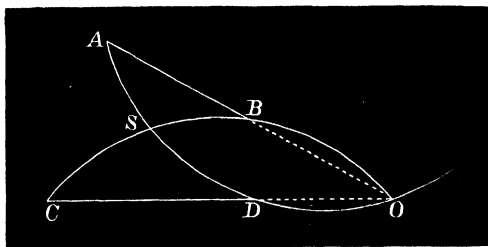
therefore, etc.

Now if  $P_1Q_1R_1$  and  $P_2Q_2R_2$  denote two fixed positions of the variable inscribed triangle  $PQR$  of constant species,

and  $PQR$  any arbitrary position, it follows that a *variable line*  $PQ$ , dividing similarly two linear segments  $P_1Q_1$  and  $P_2Q_2$ , subtends a constant angle  $POQ$  at a fixed point  $O$ .

The point  $O$  is determined by the intersection of the loci of the vertices of the triangles  $P_1Q_1O$  and  $P_2Q_2O$ , whose bases  $P_1Q_1$  and  $P_2Q_2$  are given and ratio of sides ( $=P_1P_2 : QQ_1$ ), or the intersection of the circles  $CP_1Q_1$  and  $CP_2Q_2$ .

Since  $P_1P_2$  and  $Q_1Q_2$  form similar triangles with  $O$ , this point is termed the *Centre of Similitude* of the segments. Thus the centre of similitude of two segments  $AB$  and  $CD$  is the intersection of the circles passing through the two pairs of non-corresponding extremities and the intersection  $O$  of the given lines. Or it may be regarded as the common vertex of two similar triangles described on the sides.



If the points  $B$  and  $D$  coincide,  $O$  coincides with them, and the circle  $ADO$  meeting  $CD$  in coincident points  $D$  and  $O$  therefore touches  $CD$ . In the same case the circle  $BCO$  touches  $AB$ .

COR. The centres of similitude of the sides of a triangle taken in pairs are therefore found by describing circles

on  $BC$  and  $AC$  touching the sides  $AC$  and  $BC$  respectively. The second point of intersection of these circles is a centre of similitude of  $AC$  and  $BC$ ; similarly for each of the remaining pairs of sides.

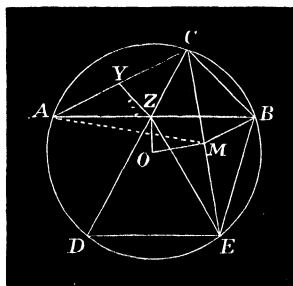
EXAMPLES.

1. Draw a line  $L$  dividing three linear segments  $A_1A_2$ ,  $B_1B_2$  and  $C_1C_2$  in the same ratio. (*Dublin Univ. Exam. Papers.*)

[Let the required line intersect the segments in  $P$ ,  $Q$  and  $R$  respectively,  $O_1$  and  $O_2$  the centres of similitude of the pairs of lines  $A_1A_2$ ,  $B_1B_2$  and  $B_1B_2$ ,  $C_1C_2$ . Then in the triangle  $O_1QO_2$  we know the base  $O_1O_2$  and vertical angle, since it is equal to  $180 - \angle_1QP - \angle_2QR$ ; therefore, etc.]

2. The centres of similitude of the sides of a triangle taken in pairs are the middle points of the symmedian chords of the circum-circle.

[Let  $X$ ,  $Y$ ,  $Z$  denote the middle points of the sides of the triangle  $ABC$ ;  $CD$  and  $CE$  the median and symmedian chords of the circle respectively;  $M$  the middle point of  $CE$ . Join  $ZE$ ,  $AM$  and  $BM$ .



Then since  $\angle ACD = \angle BCE$  and  $\angle CAZ = \angle CEB$ , the triangles  $ACZ$  and  $ECB$  are similar, and  $Y$  and  $M$  being the middle points of a pair of corresponding sides,  $CYZ$  and  $CMB$  are therefore similar. Hence  $\angle CBM = \angle CZY = \angle BCZ = \angle ACM$ . Similarly  $\angle CAM = \angle BCM$ ; therefore the triangles  $BCM$  and  $CAM$  are similar.]

3. Prove the following results from Ex. 2 :—

1°.  $CEZ = \text{difference of base angles } (B - A)$ .

2°. The triangles  $ADZ$  and  $BEZ$  equal in every respect.

3°.  $CZ \cdot CE = ab$ .

4°.  $CM = ab / \sqrt{a^2 + b^2 + 2ab \cos C}$ .

5°.  $BMC = CMA = \pi - C$ .

6°. The circum-circle of  $ABM$  passes through the centre of the circle  $ABC$ .

4. Having given the base ( $c$ ) bisector of base ( $CZ$ ) and difference of base angles ( $B - A$ ); construct the triangle.

[The triangle  $CEZ$  is readily constructed; therefore, etc.]

5. Having given the bisector of base ( $CZ$ ) rectangle under sides ( $ab$ ) and difference of base angles ( $B - A$ ); construct the triangle.

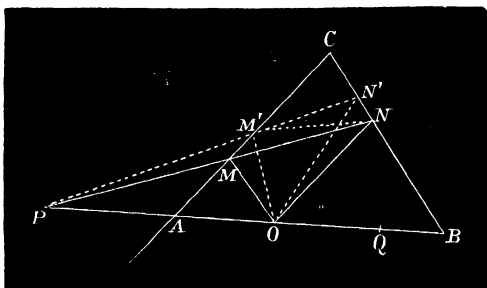
[As in Ex. 4.]

6. Having given the base, median, and symmedian of a triangle; construct it.

## SECTION IV.

### MISCELLANEOUS PROPOSITIONS.

26. **Prop. I.**—*Through a point  $P$  to draw a line across*



*an angle such that the intercepted segment  $MN$  may subtend at a fixed point  $Q$  a triangle of maximum area.*

The transversal  $PMN$  such that the parallels  $OM$  and  $ON$  to the sides of the angle intersect on  $PQ$  is the required line.

For draw any other line  $PM'N'$ . Join  $M'N$ . Then the triangles  $MON$  and  $M'ON$  are equal (Euc. I. 37), but  $M'ON > M'ON'$ ; therefore  $MON > M'ON'$ .

But  $\frac{MON}{M'ON'} = \frac{MQN}{M'QN'}$  because  $\frac{MON}{MQN}$  = ratio of the altitudes =  $PO/PQ$ . Similarly  $\frac{M'ON'}{M'QN'} = PO/PQ$ ; therefore  $MQN > M'QN'$ .

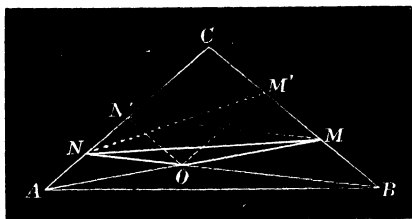
To find the point  $O$ . Evidently by similar triangles

$$PA/PO = PM/PN = PO/PB;$$

therefore

$$PA \cdot PB = PO^2.$$

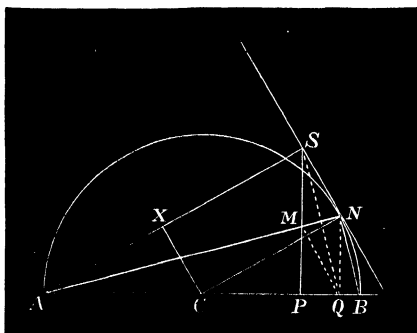
**Prop. II.**—On the sides  $BC$  and  $CA$  of a triangle, to find points  $M$  and  $N$  such that if the lines  $AM$  and  $BN$  meet in  $O$  the triangle  $MON$  may be a maximum.



Regarding  $A$  as a point on the base produced of  $BCN$  and  $AOM$  a transversal to the sides,  $MON$  is maximum when  $ON'$  and  $MN'$  parallels to these sides respectively meet on  $AC$ . Similarly since  $B$  is on the base produced of  $ACM$  and  $BON$  a transversal to the sides,  $OM'$  and  $NM'$  parallels to the sides meet on the base.

Then we have  $ANM'O$  and  $CN'OM'$  equal parallelograms (Euc. I. 36), therefore  $AN = CN'$ , also  $BM = CM'$ . But by Prop. I.  $AN \cdot AC = AN'^2$ , therefore  $AN \cdot AC = CN^2$ ; similarly  $BM \cdot BC = CM^2$ , or the sides of the triangle  $ABC$  are divided in extreme and mean ratio, the greater segments being measured from the vertex.

**Prop. III.** *Through one extremity  $A$  of the diameter  $APB$  of a semicircle draw a chord  $AMN$  to meet a perpendicular through  $P$  to the diameter  $AB$  in  $M$  and the circle in  $N$ , such that the triangle  $MBN$  may be a maximum.*



Suppose a tangent is drawn at the required point  $N$ . Let it meet  $PM$  in  $S$ . Join  $AS$ . From the centre  $C$  let fall  $CX$  perpendicular on  $AS$ . Join  $CN$ .

By Prop. I. the parallels  $MQ$  to the tangent and  $NQ$  to  $PS$  meet on  $AB$ , for then with respect to the angle  $PSN$  the triangle  $MBN$  is maximum; therefore *a fortiori* it is the maximum triangle whose vertex  $N$  lies on the circle  $ANB$ .



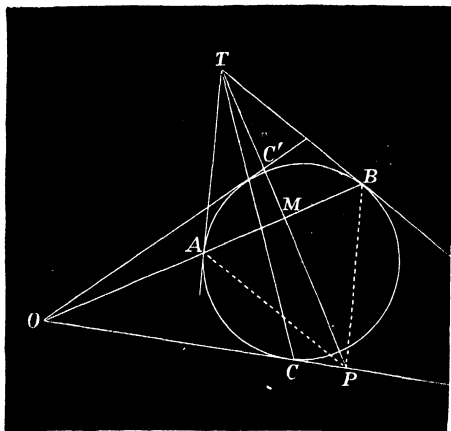


Similarly  $TT' \cdot TS = T'S^2$ , but  $TT' \cdot TS = AT^2 = TN^2$ ; therefore  $TN = T'S$ , and  $TS = T'N$ .

But when a line  $TT'$  is divided in extreme and mean ratio in  $S$  and from the greater segment a part  $T'N$  is taken equal to the less  $TS$ ,  $T'S$  is divided into extreme and mean ratio.

Ex. Draw the transversal  $AMN$  such that the quadrilateral  $MNBP$  may be a maximum.

**Prop. IV.\*** *Through a given point  $O$  in the tangent at  $C$  to a circle draw a secant  $AB$  such that the triangle  $ABC$  may be of maximum area.*



Draw tangents at  $A$  and  $B$  to meet in  $T$ . The required triangle is such that the parallels through  $A$  and  $B$  to the tangents at these points meet on  $OC$  in  $P$ .

For since  $O$  and  $T$ ,  $O$  and  $C$ , are pairs of conjugate points with respect to the circle,  $CT$  is the polar of  $O$ .

---

\* This Proposition may be omitted on the first reading.

Let  $OC'$ , the second tangent from  $O$ , meet  $PT$  in  $C'$ ,  
 Since  $PBTA$  is a rhombus,  $AB$  is at right angles to  $PT$ ;  
 also since  $TC'MP$  is a harmonic row, we have

$$TC'/C'M = TP/PM = 2;$$

therefore  $TM$  or  $PM = 3MC'$ .

Then a given angle  $COC'$  is divided by the required line  $AB$ , such that the ratio of the tangents of its segments is known; therefore, etc.

Ex. If  $\alpha$ ,  $b$ ,  $c$  denote the sides of the maximum triangle  $ABC$ ,  
 prove that

$$(1) \quad \frac{OA}{OB} = \frac{c^2 - \alpha^2}{b^2 - c^2}.$$

$$(2) \quad c^2 = \frac{\alpha^4 + b^4}{\alpha^2 + b^2}.$$

## CHAPTER III.

### RECENT DEVELOPMENTS OF POINT O THEOREM.

#### SECTION I.

#### THE BROCARD POINTS AND CIRCLE OF A TRIANGLE.

**27. Brocard Points  $\Omega$ ,  $\Omega'$ .**—In Art. 20 if the inscribed triangle  $PQR$  is similar to  $ABC$  and  $P = A$ ,  $Q = B$ ,  $R = C$ , then  $BOC = A + P = 2A$ , similarly  $COA = 2B$  and  $AOB = 2C$ ; therefore  $O$  is the centre of the circum-circle.

Secondly, let  $P = B$ ,  $Q = C$  and  $R = A$ . Then

$$BOC = A + P = A + B = \pi - C;$$

similarly  $COA = B + Q = B + C = \pi - A$ ,

and  $AOB = \pi - C$ .

Thirdly, let  $P = C$ ,  $Q = A$  and  $R = B$ . It follows as in the last case that  $BOC = \pi - B$ ,  $COA = \pi - C$  and  $AOB = \pi - A$ .

Thus we see that a triangle  $PQR$  similar to a given one may be inscribed in the latter in three different ways; and that the point  $O$  in each case may be found as in the general method by describing segments of circles on two of the sides containing given angles.

In the second and third positions the points of intersection of the circles are usually denoted by the letters  $\Omega$  and  $\Omega'$ . They are termed the *Brocard Points* of the

triangle  $ABC$ , and are distinguished as Positive ( $\Omega$ ) and Negative ( $\Omega'$ )

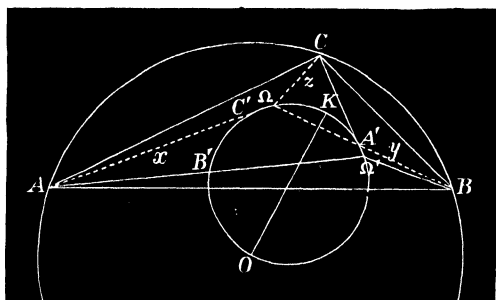
28. **Brocard Angle** ( $\omega$ ).—Since  $B\Omega C$  is the supplement of  $C$ ,  $\Omega BC + \Omega CB = C$  or  $\Omega BC = \Omega CA$ . For a similar reason  $\Omega CA = \Omega AB$ ,

hence  $\Omega BC = \Omega CA = \Omega AB = \omega$  (say).

The angle  $\omega$  is called the *Brocard Angle* of the triangle  $ABC$ .

We may remark that the angle subtended at  $\Omega$  by the base  $c$  is the supplement of  $B$ , the angle at the right extremity of  $AB$ , and at  $\Omega'$  equal to the supplement of  $A$ , the angle at the other extremity of  $AB$ .

The same relations hold for the sides  $a$  and  $b$ ; hence the names *Positive* and *Negative* Brocard points.



The value of  $\omega$  as a function of the sides or angles is thus found.

Let  $x$ ,  $y$ ,  $z$  denote the lengths of  $A\Omega$ ,  $B\Omega$  and  $C\Omega$  respectively. Then in the triangle  $B\Omega C$

$$\cot \omega = \frac{\cos \omega}{\sin \omega} = \frac{a^2 + y^2 - z^2}{2xy \sin \omega} = \frac{a^2 + y^2 - z^2}{4B\Omega C}.$$

Similarly in the triangles  $C\Omega A$  and  $A\Omega B$

$$\begin{aligned}\cot \omega &= \frac{a^2 + y^2 - z^2}{4B\Omega C} = \frac{b^2 + z^2 - x^2}{4C\Omega A} = \frac{c^2 + x^2 - y^2}{4A\Omega B} \\ &= \frac{a^2 + b^2 + c^2}{4ABC} \dots \dots \dots (1)\end{aligned}$$

It is proved in like manner for  $\Omega'$  that

$$\Omega'CB = \Omega'AC = \Omega'BA,$$

and that the value of these angles is also given by (1).

Again  $\cot A = \frac{b^2 + c^2 - a^2}{2bc \sin A} = \frac{b^2 + c^2 - a^2}{4\Delta}$  with similar values for  $\cot B$  and  $\cot C$ . Hence

$$\begin{aligned}\cot A + \cot B + \cot C &= \Sigma \frac{b^2 + c^2 - a^2}{4\Delta} = \frac{a^2 + b^2 + c^2}{4\Delta}, \\ \text{or } \cot \omega &= \cot A + \cot B + \cot C \dots \dots \dots (2)\end{aligned}$$

#### EXAMPLES.

1. Prove that

$$(1) \operatorname{cosec}^2 \omega = \operatorname{cosec}^2 A + \operatorname{cosec}^2 B + \operatorname{cosec}^2 C.$$

$$(2) \sin^2 \omega = \frac{4\Delta^2}{b^2c^2 + c^2a^2 + a^2b^2}.$$

$$(3) \cos^2 \omega = \frac{(a^2 + b^2 + c^2)^2}{4(b^2c^2 + c^2a^2 + a^2b^2)}.$$

2. The distances of  $\Omega$  from the sides of  $ABC$  are  $2R \sin^2 \omega \frac{a}{b}$ ,  $2R \sin^2 \omega \frac{a}{c}$ ,  $2R \sin^2 \omega \frac{b}{a}$ ; and of  $\Omega'$ ,  $2R \sin^2 \omega \frac{b}{c}$ ,  $2R \sin^2 \omega \frac{c}{a}$ ,  $2R \sin^2 \omega \frac{a}{b}$ .

[For let the distances of  $\Omega$  be denoted  $\alpha$ ,  $\beta$ ,  $\gamma$ . Then

$$\alpha = \gamma \sin \omega = \frac{c \sin^2 \omega}{\sin B}; \text{ therefore, etc.}$$

The ratios of the distances\* are evidently as follows:—

$$\alpha : \beta : \gamma = c^2 a : a^2 b : b^2 c,$$

and

$$\alpha' : \beta' : \gamma' = ab^2 : bc^2 : ca^2,$$

and also

$$\alpha\alpha' = \beta\beta' = \gamma\gamma' = 4R^2 \sin^4 \omega.]$$

\* Or Trilinear Co-ordinates of the points with respect to the triangle, which is also called the *Triangle of Reference*.

3.  $AD$  is the bisector of the angle  $A$  of a triangle  $ABC$ , and  $\omega_1, \omega_2$  the Brocard angles of the triangles  $ABD$  and  $ACD$  respectively; prove that  $\cot \omega_1 + \cot \omega_2 = 2 \operatorname{cosec} A + \cot A + \cot \omega$ , with similar expressions for the triangles formed by the bisectors of the angles  $B$  and  $C$ .

4. If  $\omega_1$  and  $\omega_2$  denote the Brocard angles of the triangles  $CAD$  and  $BAD$ , where  $AD$  is the median to the side  $BC$ ,

$$\cot \omega_1 - \cot \omega_2 = \frac{b^2 - c^2}{2\Delta},$$

with similar expressions for the medians  $BE$  and  $CF$ .

5. Hence prove that  $\cot \omega_1 + \cot \omega_3 + \cot \omega_5 = \cot \omega_2 + \cot \omega_4 + \cot \omega_6$ ,

$$\text{and } \Sigma \cot \omega_1 = \frac{2(a^2 + b^2 + c^2)}{\Delta}.$$

6. If  $ABC$  is divided as in the previous exercises by the symmedians, prove that  $\Sigma(b^2 + c^2)(\cot \omega_1 - \cot \omega_2) = 0$ .

7.  $\Omega$  and  $\Omega'$  are Brocard points of their pedal triangles  $PQR$  and  $P'Q'R'$ . (Euc. III. 21.)

8. The triangles  $PQR$  and  $P'Q'R'$  are equal in area.

[For  $\Omega PQ$  and  $\Omega BC$  are similar; hence (Euc. VI. 19)

$$\Omega PQ : \Omega BC = \Omega P^2 : \Omega B^2 = \sin^2 \omega;$$

similarly  $\Omega QR : \Omega CA = \Omega RP : \Omega AB = \sin^2 \omega$ ;

therefore  $PQR = P'Q'R' = ABC \cdot \sin^2 \omega$ .]

9. The Brocard points are equidistant from the circum-centre.

[By Ex. 8 and Art 23, Ex. 1.]

10. If  $A', B', C'$  be the points of intersection of the pairs of lines  $y, z' : z, x' : x, y'$ , prove that the six points  $A', B', C', O, \Omega, \Omega'$  lie on a circle.

[For the triangles  $BCA', CAB'$  and  $ABC'$  are isosceles and similar, their base angles each being equal to  $\omega$ , hence  $OA', OB', OC'$  are the bisectors of their vertical angles. In the quadrilateral  $O\Omega\Omega'A'$  we have  $O\Omega = O\Omega'$  and  $OA'$  the bisector of the angle  $\Omega A' \Omega'$ ; therefore  $O$  is a point on the circum-circle of  $\Omega A' \Omega'$ , and the quadrilateral is therefore cyclic. Similarly  $B'$  and  $C'$  are on the circum-circle of the triangle  $O\Omega\Omega'$ .]

DEF. This is called the *Brocard Circle*, and  $A'B'C'$  the *First Brocard Triangle* of  $ABC$ .

11. To find the distance of the Brocard points from the circum-centre ( $O\Omega = O\Omega' = \delta$ ).

[By Art. 23, Ex. 1,  $2PQR = (R^2 - \delta^2)\sin A \sin B \sin C$ ,

but (Ex. 8)  $PQR = ABC \sin^2 \omega = 2R^2 \sin A \sin B \sin C \sin^2 \omega$ ,

hence  $R^2 - \delta^2 = 4R^2 \sin^2 \omega$  or

$$\delta = R\sqrt{1 - 4\sin^2 \omega}.$$

12. The angle subtended at the circum-centre by  $\Omega'\Omega = 2\omega$ .

(By Ex. 10 and Euc. III. 22.)

13. To find the distance  $\Omega\Omega'$  between the Brocard points.

[Since  $O\Omega\Omega'$  is an isosceles triangle,

$$\Omega\Omega' = 2O\Omega \sin \omega = 2R \sin \omega \sqrt{1 - 4\sin^2 \omega}, \text{ by Ex. 11.}]$$

14. The diameter of the Brocard circle is equal to

$$R \sec \omega \sqrt{1 - 4\sin^2 \omega}.$$

[For it equals  $\delta/\sin 2\omega$ ; therefore, etc.]

15. The altitudes of the similar isosceles triangles  $BCA'$ ,  $CAB'$ ,  $ABC''$  are equal to the distances of the symmedian point ( $K$ ) from the sides.

$$[\text{For } C'Z = \frac{1}{2}c \tan \omega = \frac{2c\Delta}{a^2 + b^2 + c^2};$$

therefore, etc., by Art. 28, (1).]

16. The circle on  $OK$  as diameter is the Brocard circle.

[For  $KA'$  is parallel and  $OA'$  perpendicular to  $BC$ , hence  $OK$  subtends a right angle at  $A'$ ; similarly for the points  $B'$  and  $C''$ ; therefore, etc.]

17. Brocard's first triangle is *Inversely Similar* to  $ABC$ ; i.e., by rotation in the plane of the paper their sides cannot be brought into a position of parallelism with each other.

[For  $B'C''$  subtends equal angles at  $A'$  and  $K$ , but  $KB'$  and  $KC''$  are respectively parallel to  $CA$  and  $AB$ , and therefore contain an angle  $A$ ; similarly the angles  $B'$  and  $C''$  are equal to  $B$  and  $C$ .]

18. Having given the base  $c$  and Brocard angle  $\omega$  of a triangle  $ABC$ , find the locus of the vertex (Neuberg).

[Let  $\rho$  be the median  $CZ$  and  $\theta$  the angle between it and  $PZ$ . Since  $\cot \omega = (a^2 + b^2 + c^2)/2c \cdot CR$  and  $a^2 + b^2 = \frac{1}{2}c^2 + 2\rho^2$ , we have

$$2\rho^2 + \frac{3}{2}c^2 = 2c \cot \omega \cdot CR = 2c \cot \omega \cdot \rho \cos \theta,$$

$$\text{or} \quad \rho^2 - c \cot \omega \cdot \rho \cos \theta + \frac{3}{4}c^2 = 0.$$

NOTE.—Comparing this result with the standard form of the equation in the footnote we have by equating coefficients

$$c \cot \omega = 2d \text{ and } d^2 - r^2 = \frac{3}{4}c^2,$$

$$\text{or} \quad d = \frac{1}{2}c \cot \omega \text{ and } r^2 = \frac{1}{4}c^2 \cot^2 \omega - \frac{1}{4}c^2.$$

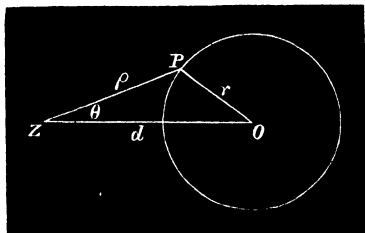
It is evident that the locus is a curve symmetrical with respect to the perpendicular bisector of the base, as to each position of the vertex  $C$  there is a corresponding one,  $C''$  of the inversely similar triangle  $ABC''$  described on the base.

The distance of  $C'$ , a vertex of Brocard's first triangle, from  $c = \frac{1}{2}c \tan \omega$ ; therefore  $ZC' \cdot ZO = (\frac{1}{2}c)^2$  where  $O$  is the centre of the required locus.

This example is a particular case of :—*Having given the base  $c$  and  $(la^2 + mb^2 + nc^2)/\Delta$  to find the locus of the vertex*; a solution of which is similarly obtained.

18a. Six similar triangles are constructed on a given base and on the same side of it. Prove that their vertices  $C_1, C_2, \dots, C_6$  are concyclic. (Mathesis, t. 2, p. 94.)

\* This is known by Analytical Geometry to be the *Polar Equation of a Circle*. If we take any point  $Z$  and draw a variable line (*Radius Vector*) to a given circle ( $O, r$ ) and let  $d = ZO$ , the equation connecting  $\rho$  and  $\theta$



is for all points on the circle  $\rho^2 - 2\rho d \cos \theta + d^2 - r^2 = 0$ ; and  $\rho$  and  $\theta$  are called the *Polar Co-ordinates* of the point  $P$ .





Again from (1)  $AZ'/c = b^2/(a^2 + b^2)$  or  $AZ' = b^2c/(a^2 + b^2)$  ;  
 similarly  $BZ' = a^2c/(a^2 + b^2)$ .....(2)

Also  $CZ'/CK = a'/a = (a^2 + b^2 + c^2)/(a^2 + b^2)$ , hence

$$\frac{CK}{KZ'} = \frac{a^2 + b^2}{c^2} \dots\dots\dots(3)$$

COR. If  $C = 90^\circ$  then  $CK = KZ'$  (Euc. I. 47) and  $K$  is the middle point of the perpendicular on the hypotenuse.

30. The length of the symmedian  $CZ'$  is found as follows :—

In the formula  $b^2BZ' + a^2AZ' = cAZ' \cdot BZ' + cCZ'^2$  substitute the values in (2) and reduce. We easily obtain

$$CZ' = \frac{\sqrt{a^2 + b^2 + 2ab \cos C}}{a/b + b/a}$$

with similar expressions for the lines through  $A$  and  $B$ .

#### EXAMPLES.

1. The symmedian is divided harmonically at  $K$ , and  $Q$  its point of intersection with the perpendicular to the base of the triangle at its middle point  $Z$ .

$$\left[ ZZ' = \frac{b^2c}{a^2 + b^2} - \frac{c}{2} = \frac{c^2}{a^2 + b^2}ZR; \text{ hence} \right. \\
 \frac{CP}{ZZ'} = \frac{ZR}{ZZ'} = \frac{a^2 + b^2}{c^2} = \frac{CK}{KZ'}, \text{ (Art. 29 (3));} \\
 \left. \text{therefore } CQ/QZ' = CK/KZ' = (a^2 + b^2)/c^2. \right]$$

2. Since  $Z, CKZ'Q$  is an harmonic pencil any line through  $K$  is cut harmonically by its rays, hence if  $KC'$  is parallel to one ray, it is bisected at  $O$  by the conjugate ray  $CZ$ . Also the parallel through  $K$  to  $PL$  is bisected at  $K$ .

3. The vertices of Brocard's first triangle and the symmedian point are equidistant from the extremities of the parallels through  $K$  to the sides of  $ABC$ .

[Let  $O$  be the middle point of  $MN$ . Since  $OM = ON$  and (Ex 2)  $OK = OC'$ , subtracting these results ; therefore, etc.]

4. The lines joining the middle points of the sides of  $ABC$  to the middle points of the perpendiculars on them meet in a point.

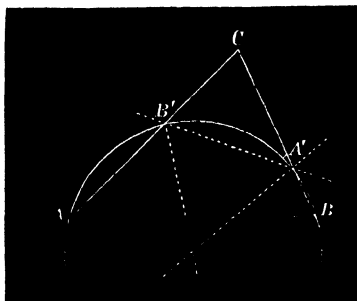
[By Ex. 2 the point of concurrence is the symmedian point. The ratios of the segments into which the joining lines are divided at  $K$  are easily seen to be  $bc \cos A/a^2$ , etc., etc.]

5. Prove that  $\cot KBC + \cot KCA + \cot KAB = 3 \cot \omega$ .

6. The sides of the pedal triangle of  $K$  are at right angles to the medians of  $ABC$ .

### ANTIPARALLELS.

**Def.** A straight line meeting the sides  $a$  and  $b$  of a triangle at angles  $A$  and  $B$  is parallel to the base. If a line meet these sides at angles  $A$  and  $B$  respectively it is said to be *Antiparallel* to  $c$ .



31. The following are the fundamental and obvious properties of antiparallels to the sides of any triangle:—

(1) Antiparallels to the sides  $a$  and  $b$  meet  $c$  at equal angles ( $C$ ).

(2) They are parallels to the sides of the pedal triangle.

(3) Or to the tangents at  $A$ ,  $B$ ,  $C$  to the circum-circle.

(4) The locus of the middle point of a variable anti-parallel to a side,  $c$ , is the corresponding symmedian chord  $CK$ .

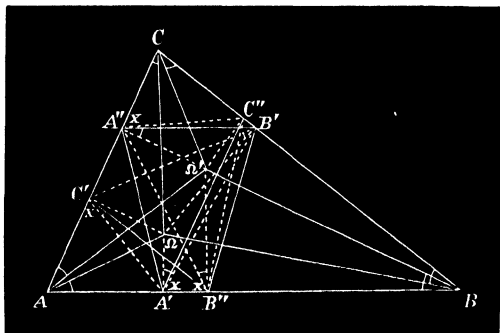
(5) Antiparallels through  $K$  to each side are bisected at the point, and are equal to one another. The latter part follows from (1).

(6) The median and symmedian to  $c$  of the triangle  $ABC$  are respectively the symmedian and median of the triangle  $A'B'C$  cut off by any antiparallel  $A'B'$ .

(7) The extremities of a parallel and antiparallel to any side of a triangle are concyclic.

### THE PEDAL TRIANGLES OF THE BROCARD POINTS.

32. From  $\Omega$  let fall perpendiculars on the sides and denote their feet as in figure by  $A'B'C'$ .



It follows conversely since  $A\Omega B$  is the supplement of  $B$  (Art. 28), and is equal to  $C + A'$  (Art. 19) that  $A' = A$ ; similarly  $B' = B$  and  $C' = C$ . Also  $A''$ ,  $B''$ ,  $C''$  are respectively equal to  $A$ ,  $B$  and  $C$ .

**33. Theorems** I.  $\Omega$  is the common positive Brocard point of  $ABC$  and  $A'B'C'$ .

Since  $AC'A'\Omega$  is a cyclic quadrilateral  $\Omega AB = \Omega C'A' = \omega$  (Euc. III. 21); similarly  $\Omega B'C'$  and  $\Omega A'B'$  are each equal to  $\omega$ .

It follows also that  $\Omega'$  is the common negative Brocard point of  $ABC$  and  $A''B''C''$ .

II. The sides of  $A'B'C'$  and  $A''B''C''$  are equally inclined to the corresponding sides of  $ABC$ .

For by (1)  $CB'C' = AC'A' = BA'B' = 90 - \omega$ ,  
and  $BC''B'' = AB''A'' = CA''C'' = 90 - \omega$ .

III. The six points  $A', B', C', A'', B'', C''$  are concyclic.

For the angles  $AC'A' = AB''A''$ , therefore  $A'A''B''C'$  is cyclic (Euc. III. 22).

Similarly  $B'B''C''A'$  and  $C'C''A''B'$  are cyclic. But generally if three pairs of points on the sides of a triangle are such that every two pairs are cyclic, the six points lie on a circle.\* For if they do not the tangents to the three circles from  $A, B$  and  $C$  are easily seen to be equal, which is impossible.

IV. The lines  $B'C', C''A', A''B'$  are parallel to the sides  $a, b, c$  respectively.

We know that each pair of sides of  $ABC$  with  $\Omega$  and  $\Omega'$  form similar triangles, i.e.,  $B\Omega C$  and  $A\Omega'C$ ,  $C\Omega A$  and  $B\Omega'A$ ,  $A\Omega B$  and  $C\Omega'B$  are similar; hence the perpendiculars (or other corresponding lines) through  $\Omega$  and  $\Omega'$  divide the opposite sides similarly. In the triangles  $C\Omega A$

---

\* For example, if  $A'B'C'$  be the middle points of the sides and  $A''B''C''$  the feet of the perpendiculars, it follows immediately that  $A'B'C'A''B''C''$  is a cyclic hexagon since each pair of points  $AA'$  and  $BB'$  form a cyclic quadrilateral. ("Nine Points" Circle.)

and  $B\Omega A$  we have therefore  $AC'/AC = AB''/AB$ , or  $B''C'$  is parallel to  $a$ .

V. Hence also  $A'A''$ ,  $B'B''$ ,  $C'C''$  are antiparallels to the sides  $a$ ,  $b$ ,  $c$ . (Euc. III. 22.)

### SECTION III.

#### TUCKER'S CIRCLES.

34. By Art. 24 if the inscribed triangle  $A'B'C'$  is given in species only it may be conceived to vary its position by rotating around the point  $\Omega$  which is fixed. Let it revolve in a positive direction through any angle  $\theta$  and also let  $A''B''C''$  revolve in the opposite direction through an equal angle.

Then each of the equal angles of inclination of the sides of  $A'B'C'$  and  $A''B''C''$  are diminished by  $\theta$ , therefore for all values of  $\theta$  the sides are equally inclined and the vertices of the two triangles are always concyclic.

The circles thus described are called the *Tucker Circles* of the triangle.

Thus the lines  $B''C'$  and  $A'A''$ , etc., are always parallel and antiparallel respectively to the opposite side  $a$ , and therefore remain constant in direction.

Now since the point  $\Omega$  is fixed and the triangle  $A'B'C'$  of constant species; since the vertices move on given lines all points fixed relatively to the figure describe lines. The locus of the centre of the system of Tucker's circles is therefore a line. (Art. 20.)

By taking particular positions of the triangle we find points on the line of centres. In the case where  $\theta=0$  the

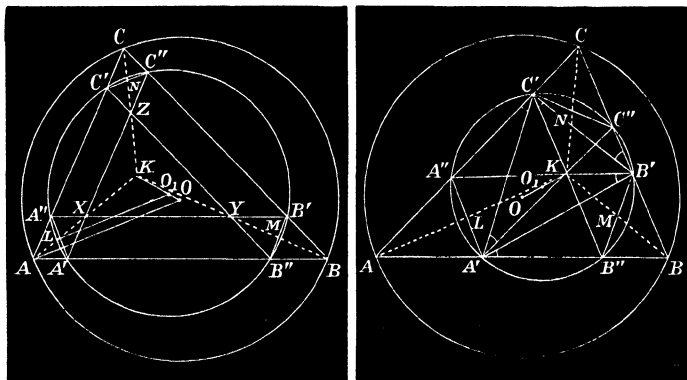
vertices of  $ABC$  and  $A'B'C'$  coincide, and the circum-circle is thus seen to be one of Tucker's circles. The line of centres thus passes through the circum-centre of  $ABC$ .

Similarly the loci of the other Brocard points of the triangle  $A'B'C'$  and  $A''B''C''$  are lines.

35. Let the vertices of the triangle formed by the parallels  $B''C'$ ,  $C''A'$ ,  $A''B'$  to the sides of  $ABC$  be denoted by  $X$ ,  $Y$ ,  $Z$ .

Then  $AA'A''X$  is a parallelogram, as are also  $BB'B''Y$ ,  $CC'C''Z$ ; and since the diagonals bisect each other  $AX$  bisects the antiparallel  $A'A''$ .  $AX$ ,  $BY$ ,  $CZ$  are the symmedians of  $ABC$ .

Hence the following construction for Tucker's circles :



Let  $K$  be the symmedian point of  $ABC$ . Join  $AK$ ,  $BK$ ,  $CK$ . Take any point  $X$  on  $AK$  and draw parallels through it to the sides  $b$  and  $c$ . Let them meet  $BK$  and  $CK$  in  $Y$  and  $Z$  respectively.  $YZ$  is parallel to  $a$ , and the hexad of points in which the sides of  $ABC$  are cut by these parallels lie on one of the required circles.

36. *The antiparallels  $A'A''$ ,  $B'B''$ ,  $C'C''$  are equal.*  
For since  $A''B'$  is parallel to  $c$ , and  $A'A''$  and  $B'B''$  are equally inclined to  $c$  (at an angle  $C$ ),  $A'A'' = B'B''$ ; therefore, etc.; or they are the chords of a Tucker circle intercepted by parallel lines.

37. **Theorem.** *The line  $OK$  is the locus of the centre of Tucker's system of circles.*

For let  $L$  be the middle point of the chord  $A'A''$  of one of the system. Draw  $LO_1$  at right angles to it meeting  $OK$  in  $O_1$ . Join  $AO$ .

Since the tangent at  $A$  to the circum-circle is antiparallel to  $a$ ,  $AO$  and  $LO_1$  are parallel lines.

But  $AK/AX = BK/BY = CK/CZ$  (Euc. VI. 2); therefore  $AK/AL = BK/BM = CK/CN = OK/OO_1$ , or  $O_1$  is the centre of the Tucker circle.

38. Since  $\Omega$  is the positive Brocard point of the triangles  $ABC$  and  $A'B'C'$ , and  $\Omega AB$  and  $\Omega A'B'$  a pair of similar triangles; if  $\theta$  be the inclination of the sides of  $A'B'C'$  to those of  $ABC$ , we have

$$\frac{\Omega A'}{\Omega A} = \frac{\sin \omega}{\sin(\theta + \omega)} \dots \dots \dots (1)$$

This ratio is the *Ratio of Similitude* of the triangles, and is the constant relation between all corresponding lines of  $A'B'C'$  and  $ABC$ .

For example, if  $\rho$  be the radius of Tucker's circle for any value of  $\theta$ ,

$$\frac{\rho}{R} = \frac{\sin \omega}{\sin(\theta + \omega)} \dots \dots \dots (2)$$

In (2) we have the following particular cases:—

when  $\theta = 0^\circ$        $\rho = R \dots \dots \dots$  (circum-circle);

„     $\theta = \omega$        $\rho = \frac{1}{2} R \sec \omega \dots \dots \dots$  (T. R. circle);

„     $\theta = 90^\circ$      $\rho = R \tan \omega \dots \dots \dots$  (cosine circle).

Also area  $A'B'C' : ABC = \sin^2 \omega : \sin^2(\theta + \omega)$  (Euc. VI. 19).



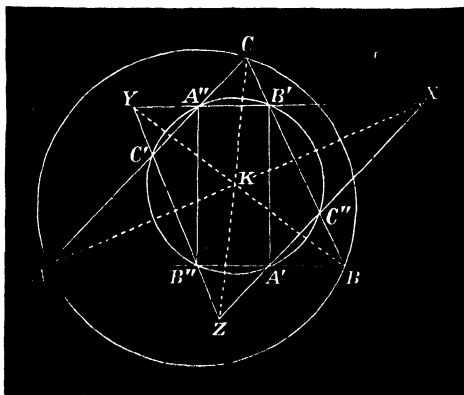
## SECTION IV.

## TUCKER'S CIRCLES, PARTICULAR CASES.

39. I. **Cosine Circle.** As a particular case of the general theorem (Art. 33 v.) we shall consider the antiparallels  $A'A''$ ,  $B'B''$ ,  $C'C''$  to pass through  $K$ . The points  $L$ ,  $M$ ,  $N$  will therefore coincide with  $K$ , which is also the centre of the corresponding Tucker's circle.

It is otherwise evident that the six segments  $KA'$ ,  $KA''$ , etc., of antiparallels through  $K$  to the sides are equal. (Art. 31 (5)).

Also  $B'C'B''C''$ ,  $C'A'C''A''$ ,  $A'B'A''B''$  are rectangles since their diagonals are equal.



Again because  $A'B'B''$  is a right-angled triangle

$$A'B'' = B'B'' \cos A'B'B'' = B'B'' \cos C,$$

or  $A'B'' = 2\rho \cos C$ , with similar expressions for  $B'C''$  and  $C'A''$ . Hence

*The segments intercepted by the circle on the sides of  $ABC$  are proportional to the cosines of the opposite angles.\**

It is from this property the circle derives its name.

40. The middle point  $M$  of  $A''B'$  is on the median through  $C$  to the opposite side  $c$ ; hence the perpendicular through  $K$  to this side passes through  $M$ , or as has been shown otherwise (Art. 30, Ex. 4). If a perpendicular be drawn through  $K$  to the base meeting it in  $N$  and the median in  $M$ ,  $MK = NK$ , from which it follows *that the lines joining the middle points of the sides to the middle points of the corresponding perpendiculars meet at the symmedian point (Hain).*

41. The sides of the triangles  $A'B'C'$  and  $A''B''C''$  are perpendicular to the corresponding sides of  $ABC$ . The cosine circle may therefore be obtained by rotating the two inscribed triangles in opposite directions until  $\theta = 90^\circ$ . (Art. 39.)

The ratio of similitude of  $A'B'C'$  and  $ABC = \tan \omega$ .

42. II. **Triplicate Ratio Circle.**—Let the parallel in figure of Art. 35 pass through  $K$ .

Then  $L, M, N$  are the middle points of  $AK, BK$ , and  $CK$ , since  $AA'A''K$ , etc., etc., are parallelograms; and the centre  $O$  of the corresponding Tucker circle bisects  $OK$ .

The sides of  $A'B'C'$  are inclined to those of  $ABC$  at an angle  $= \omega$ . For consider the angles in the equal segments  $A'A''$ ,  $B'B''$ ,  $C'C''$ , and it is obvious (Euc. III. 21) that  $A'B'A'' = A'C'A'' = B'C'B'' = B'A'B'' = C'A'C'' = C'B'C''$ .

\* See *Mathesis*, t. i., p. 185 :—

“Sur le centre des Médiannes Antiparallèles,” Neuberg (1881).

Hence  $K$  is the negative Brocard point of  $A'B'C'$ .

Similarly it is the positive Brocard point of  $A''B''C''$ .

It follows generally that the locus of the negative Brocard point of  $A'B'C'$  is a line passing through  $K$ .

43. The ratio of similitude of  $A'B'C'$  and  $ABC$  is  $\sin \omega / \sin 2\omega$  since  $\theta = \omega$ ; hence

$$\rho = \frac{1}{2}R \sec \omega \dots \dots \dots (1)$$

44. The intercepts  $B'C''$ ,  $C'A''$ ,  $A'B''$  made by the circle on the sides are thus determined:—The triangles  $A'KB''$  and  $ABC$  are similar, therefore  $A'B''/c$  = ratio of

$$\text{altitudes} \quad = \frac{2c\Delta}{a^2 + b^2 + c^2} \bigg/ \frac{2\Delta}{c} = \frac{c^2}{a^2 + b^2 + c^2};$$

$$\text{hence} \quad A'B'' = \frac{c^3}{a^2 + b^2 + c^2} \dots \dots \dots (1)$$

with similar expressions for  $B'C''$  and  $C'A''$ . The general property of the circle may be thus stated:—*Parallels through the symmedian point meet the non-corresponding sides in six points which lie on a circle*; and the intercepts made on each side are in the ratios  $a^3 : b^3 : c^3$ . From the latter property the circle takes its name. For the sake of brevity it is often written "T.R." Circle.\*

45. III. **Taylor's Circle.**—Let the antiparallels  $A'A''$ ,  $B'B''$ ,  $C'C''$ , which, it will be remembered, are always parallel to the sides of the pedal triangle ( $PQR$ ) of  $ABC$ , pass through the middle points  $\alpha$ ,  $\beta$ ,  $\gamma$  of the sides of  $PQR$ .

Consider the segments into which  $A'A''$  is divided by  $\beta$  and  $\gamma$ . We have  $\beta\gamma = \frac{1}{2}QR$ ,  $\gamma A'' = \frac{1}{2}PQ$  (Euc. I. 5), and

---

\* An account of the circle will be found in *Mathesis* in the article by Neuberg already referred to (Art. 39). See also *Nouvelles Annales*, 1873, p. 264.

for the same reason  $\beta A' = \frac{1}{2}RP$ ; therefore  $A'A''$  is equal to the semiperimeter of  $PQR$

$$= \frac{1}{2}(a \cos A + b \cos B + c \cos C) = 2R \sin A \sin B \sin C.$$

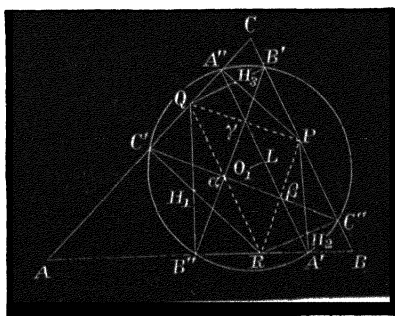
Hence generally

$$A'A'' = B'B'' = C'C'' = 2R \sin A \sin B \sin C \dots (1)$$

Again, since  $B''aC'$  is an isosceles triangle, the perpendicular to the chord  $B''C'$  of Tucker's circle at the middle point bisects the vertical angle  $a$  and passes through the in-centre of  $a\beta\gamma$ . Similarly for the chords  $C''A'$  and  $A''B'$ . Hence

*The centre of the circle coincides with the in-centre of the median triangle ( $a\beta\gamma$ ) of  $PQR$ .*

Many properties of this circle are proved in Neuberg's article in *Mathesis*, t. 1, p. 185, but it was described independently in England by Mr. H. M. Taylor, and now bears his name. (*Proc. Lond. Math. Society*, vol. xv. p. 122.)



46. Since  $aQ = aR = aB'' = aC''$ , the circle on  $QR$  as diameter passes through  $B''$  and  $C''$  and  $RB''Q = RC''Q = 90^\circ$ ; or  $B''$  and  $C''$  are the projections of  $Q$  and  $R$  on the sides  $AB$  and  $AC$ ; hence

*The six projections of the vertices of the pedal triangle on the sides of  $ABC$  lie on Taylor's circle.*

47. The triangle  $B'aC''$  is isosceles, therefore  $O_1\alpha$  the bisector of its vertical angle  $\alpha$  is at right angles to  $BC$ ; hence generally

*The lines  $O_1\alpha$ ,  $O_1\beta$ ,  $O_1\gamma$  are perpendiculars to the sides of  $ABC$ .*

Let  $H_3$  denote the orthocentre of  $CPQ$ ; then  $QH_3$  and  $O_1\alpha$  are parallel; similarly  $PH_3$  and  $O_1\beta$  are parallel; hence the triangles  $PQH_3$  and  $\alpha\beta O_1$  are similar, their ratio of similitude being  $=\frac{1}{2}$ , or  $H_3R$  is bisected at  $O_1$ .

Similarly  $PH_1$  and  $QH_2$  are each bisected at  $O_1$ ; and therefore the triangles  $H_1H_2H_3$  and  $PQR$  are equal in all respects.

48. **Theorem.**—*Taylor's circle of the triangle  $ABC$  is the common orthogonal circle of the ex-circles of  $PQR$ .*

In the triangle  $AA'A''$  we have by rule of sines

$$AA'' = A'A'' \sin C / \sin A = 2R \sin B \sin^2 C \quad (\text{Art. 45 (1)}),$$

also  $AC' = AR \cos A = b \cos^2 A$ ;

multiplying these results and reducing

$$AA'' \cdot AC' = 4R^2 \sin^2 B \sin^2 C \cos^2 A,$$

but  $AQ = c \cos A$ ; substituting we obtain

$$AA'' \cdot AC' = AQ^2 \sin^2 B,*$$

or the square of the perpendicular from  $A$  on  $QR$ . Hence the tangent from  $A$  an ex-centre of  $PQR$  to Taylor's circle

\* Otherwise from the right-angled triangle  $AA''P$  and  $ACP$  we have

$$AA'' = b \sin^2 C; \text{ and from the triangles } ACR \text{ and } AC'R,$$

$$AC' = b \cos^2 A; \text{ therefore } AA'' \cdot AC' = b^2 \sin^2 C \cos^2 A.$$

is equal to the radius of the ex-circle; similarly for the ex-centres  $B$  and  $C$ ; therefore, etc.\*

# EXAMPLES.

1. To find the value of the radius  $\rho$  of a circle cutting the ex-circles of a triangle  $PQR$  orthogonally.

[In figure of Art. 45  $\rho^2 = O_1A'^2$ . But if a perpendicular be drawn from  $O_1$  to  $\beta\gamma$  it is equal to the radius of the in-circle of the triangle  $\alpha\beta\gamma$  or half the radius ( $\frac{1}{2}r$ ) of  $PQR$ ; and the distance of its foot from  $A'$  is equal to the semiperimeter of  $\alpha\beta\gamma$ —i.e.,  $\frac{1}{2}s$  of  $PQR$ .

Hence (Euc. I. 47)  $\rho^2 = \frac{1}{4}(r^2 + s^2)$ .

Similarly for the radii  $\rho_1, \rho_2, \rho_3$  of the circles cutting two escribed and the inscribed of  $PQR$  orthogonally we obtain

$$\rho_1^2 = \frac{1}{4}(r_1^2 + s - a^2),$$

$$\rho_2^2 = \frac{1}{4}(r_2^2 + s - b^2),$$

$$\rho_3^2 = \frac{1}{4}(r_3^2 + s - c^2),$$

and by adding these results we have, on reducing,

$$\rho^2 + \rho_1^2 + \rho_2^2 + \rho_3^2 = 4R^2$$

or,

*the sum of the squares of the radii of the four circles cutting orthogonally the inscribed and escribed circles of any triangle taken in threes is equal to the square of the diameter of the circum-circle.*

---

\* In the triangle  $PQR$  since perpendiculars  $PA'$  and  $QB''$  are let fall from the extremities of the base  $PQ$  on the external bisector  $AB$  of the vertical angle  $R$ , by a well-known property  $\gamma A' = \gamma B'' = \frac{1}{2}$  sum of sides. But the distance of the middle point of any side from the points of contact of the ex-circles which touch it externally  $= \frac{1}{2}$  sum of sides. Hence if a circle be described with  $\gamma$  as centre and  $\gamma A' = \gamma B''$  as radius, it cuts the ex-circles of  $PQR$  whose centres are at  $A$  and  $B$  orthogonally. It follows that the locus of the centre of a circle cutting these two orthogonally is the line  $\gamma O_1$ , since it is perpendicular to the line of centres; similarly  $\alpha O_1$  and  $\beta O_1$  are the loci for the centres of circles orthogonal to the remaining pairs of ex-circles, whose centres are at  $B$  and  $C$ ,  $C$  and  $A$  respectively.

Therefore  $O_1$  is the centre and  $O_1A' = O_1B'' = \text{etc.}$ , the radius of the common orthogonal circle, i.e., Taylor's circle.

2. To find the radius  $\rho$  of Taylor's circle of a triangle  $ABC$ .

[Taylor's circle for the triangle  $ABC$  is the circle in Ex. 1 for  $PQR$ ; hence we have to express  $r$  and  $s$  of the latter triangle\* in terms of the parts of  $ABC$ . We easily obtain

$$\rho^2 = 4R^2(\sin^2 A \sin^2 B \sin^2 C + \cos^2 A \cos^2 B \cos^2 C)$$

$$\text{also } \rho_1^2 = 4R^2(\sin^2 A \cos^2 B \cos^2 C + \cos^2 A \sin^2 B \sin^2 C)$$

with similar values for  $\rho_2^2$  and  $\rho_3^2$ .

From these expressions we have the result given in Ex. 1:  $\Sigma \rho^2 = 4R^2$ .]

3. The lines  $B''C'$ ,  $C''A'$ ,  $A''B'$ , parallels to the sides of  $ABC$ , are the chords of contact of the ex-circles of  $PQR$  with its sides.†

[Let  $A''B'$  meet  $PR$  in the point  $Q'$ . Then  $B'B''RQ'$  is a parallelogram, therefore  $RQ' = \text{semiperimeter of } PQR$ , etc.]

4. Employing the notation of Art. 35, prove that the lines joining the corresponding vertices of the two triangles  $PQR$  and  $XYZ$  are concurrent at the circum-centre of the latter.

[Let  $p$  and  $q$  be the perpendiculars from  $R$  on the sides  $YZ$  and  $ZX$  of the triangle  $XYZ$ . Then  $p/q = RB'/RA' \sin B / RA' \sin A$ . But  $RB'/RA' = QR/RP = a \cos A / b \cos B$ . Substituting and reducing we have  $p/q = \cos A / \cos B$ .

But if  $Z$  be joined to the circum-centre of  $XYZ$ , the joining line is the locus of a point such that perpendiculars from it on the sides are in this ratio; hence  $ZR$  passes through the circum-centre of  $XYZ$ .‡ And similarly for the lines  $PX$  and  $QY$ .]

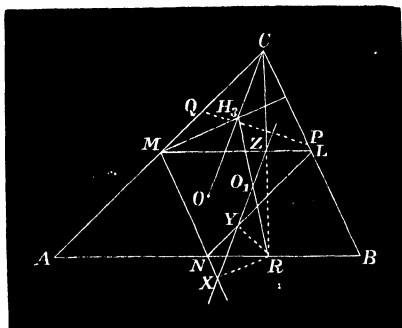
\* The sides of the pedal triangle are equal to  $a \cos A$ ,  $b \cos B$ ,  $c \cos C$ , or  $R \sin 2A$ ,  $R \sin 2B$ ,  $R \sin 2C$ ; hence its perimeter  $= 4R \sin A \sin B \sin C$ ; its  $s - a = 2R \sin A \cos B \cos C$ , its  $s - b = 2R \cos A \sin B \cos C$ , etc.; its  $r = 2R \cos A \cos B \cos C$ ; its  $r_1 = 2R \cos A \sin B \sin C$ , etc.

† The polars of the vertices of a triangle with respect to the ex-circles meet the sides in six points which lie on the same circle.—*Mathesis*, t. 1, p. 190.

‡  $PX$ ,  $QY$ , and  $RZ$  are perpendiculars to antiparallels to the sides of  $XYZ$  and therefore meet the sides of  $PQR$  at right angles.

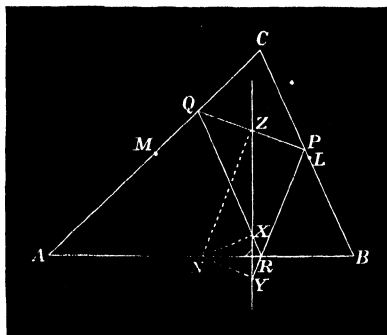
Hence the circum-centre of  $XYZ$  is the orthocentre of the triangle  $PQR$ .

5. The Simson lines of the median triangle  $LMN$  of a given one  $ABC$  with respect to the vertices  $P, Q, R$  of the pedal triangle pass through the centre of Taylor's circle.\*



[The circum-centre  $O$  of  $ABC$  is the orthocentre of  $LMN$ . Hence  $RO$  is bisected by the Simson line  $XYZ$  of  $R$ . Also  $CZ=RZ$ ; therefore the line  $XYZ$  is parallel to  $OC$ . But the centre of Taylor's circle  $O_1$  is (Art. 47) the middle point of  $RH_3$ ; therefore, etc.]

6. The Simson lines of  $PQR$ , whose poles are  $L, M, N$ , pass through  $O_1$ .



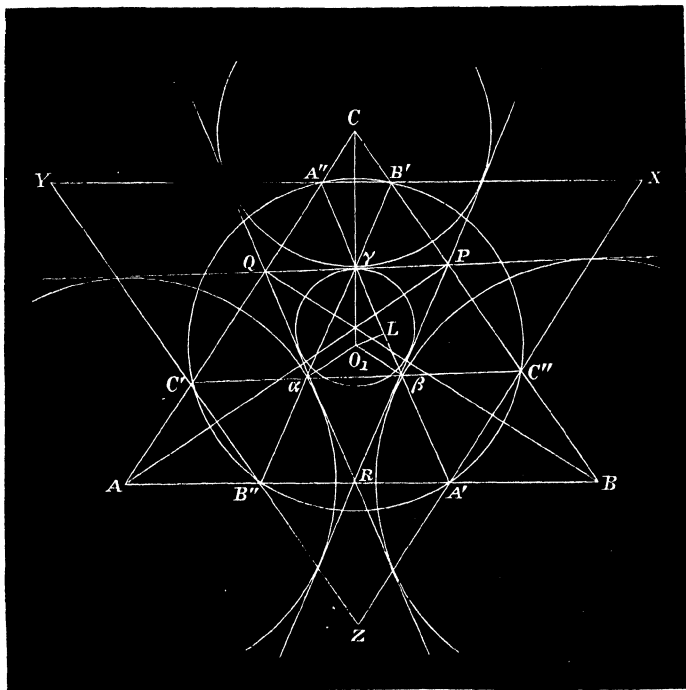
\* The point on the circum-circle from which perpendiculars or other isoclinals are let fall on the sides of an inscribed triangle is called the *Pole* of the Simson line.—*V. Mathesis*, t. 2, p. 106, "Sur la Droite de Simson," par M. Barbarin.



[For the perpendicular  $NZ$  from  $N$  on  $PQ$  bisects it (Euc. III. 3); and the perpendiculars  $NX$  and  $NY$  are equally inclined to  $AB$  (Euc. I. 26), hence the line  $XYZ$  is a perpendicular to  $AB$  through the middle point of  $PQ$ ; therefore, etc. (Art. 47.)]

7. Prove that the common inclination ( $\theta$ ) of the sides of the triangles  $A'B'C'$  and  $A''B''C''$  to those of  $ABC$  is given by the equation

$$\tan \theta = -\tan A \tan B \tan C. \quad (\text{Taylor})$$



8. The intercepts made by Taylor's circle on the sides are  $a \cos A \cos (B - C)$ ,  $b \cos B \cos (C - A)$ ,  $c \cos C \cos (A - B)$ .

$$[A'B' = A'R + RB' = (a \cos A + b \cos B) \cos C = \text{etc.}]$$

9. The circum-centre of a triangle, its symmedian point, and the orthocentre of its pedal triangle are collinear. (Tucker.)

[The orthocentre of the pedal triangle has been shown to be (Ex. 4) the circum-centre of  $XYZ$ , and  $K$  is the centre of similitude of  $ABC$  and  $XYZ$ ; therefore, etc.]

10. The circum-centre and the orthocentre of its pedal triangle are equidistant from, and collinear with, the centre of Taylor's circle. (Neuberg.)

[For  $CH_3$  and  $ZR$  are parallel, since both are at right angles to  $PQ$ ; also  $RH_3$  is bisected at  $O_1$  (Art. 47), therefore, etc., by Art. 37.]

## CHAPTER IV.

### GENERAL THEORY OF THE MEAN CENTRE OF A SYSTEM OF POINTS.

49. We now proceed to the discussion of the general linear relation connecting the distances of a system of points from a given line.

Let  $A, B, C, D \dots$  be the system of points,  $AL, BL, CL \dots$  their distances from any line  $L$ , and  $\Sigma(a \cdot AL)$  the algebraic sum

$$a \cdot AL + b \cdot BL + c \cdot CL + \dots$$

where  $a, b, c \dots$  are given quantities.

By  $\Sigma(a \cdot AL)$  is therefore meant the sum of given multiples of the distances of the system of points from the line; perpendiculars from points on opposite sides of  $L$  being taken with opposite signs.

50. **Theorem.**—*For any two lines  $M$  and  $N$  and systems of points  $A, B, C \dots$  and multiples  $a, b, c \dots$  having given*

$$\Sigma(a \cdot AM) = 0 \text{ and } \Sigma(a \cdot AN) = 0$$

*to prove that*

$$\Sigma a \cdot AL = 0,$$

*where  $L$  is any line passing through  $O$  the intersection of  $M$  and  $N$ .*

Join  $AO$  and let this line be denoted by  $R$ . Then since  $LMNR$  is a concurrent system of lines we have

$$\sin MN \cdot \sin LR + \sin NL \cdot \sin MR + \sin LM \cdot \sin NR = 0,$$

but, by Art. 2,

$$\sin LR : \sin MR : \sin NR = AL : AM : AN;$$

therefore

$$\sin MN \cdot AL + \sin NL \cdot AM + \sin LM \cdot AN = 0.$$

Similarly for the points  $B, C \dots$  we have

$$\sin MN \cdot BL + \sin NL \cdot BM + \sin LM \cdot BN = 0$$

$$\sin MN \cdot CL + \sin NL \cdot CM + \sin LM \cdot CN = 0.$$

Multiplying these equations respectively by  $a, b, c \dots$  and adding

$$\sin MN \Sigma(a \cdot AL) + \sin NL \Sigma(a \cdot AM) + \sin LM \Sigma(a \cdot AN) = 0,$$

hence if  $\Sigma(a \cdot AM) = 0$  and  $\Sigma(a \cdot AN) = 0$ , it follows that

$$\Sigma a \cdot AL = 0.$$

**Def.** The point  $O$  which satisfies the relation  $\Sigma(a \cdot AL) = 0$  for every line  $L$  passing through it is termed the *Mean Centre* of the system of points  $A, B, C \dots$  for the system of multiples  $a, b, c \dots$

**51. Theorem.**—*The position of the mean centre for a given system of multiples is either unique or indeterminate.*

For let  $O_1$  and  $O_2$  be two of its positions, and  $O$  any point whatever. Join  $O_1O$  and  $O_2O$ , and denote these lines by  $M$  and  $N$ .

Since  $\Sigma(a \cdot AM) = 0$  and  $\Sigma(a \cdot AN) = 0$ , it follows by Art. 50 that any line  $L$  through  $O$ , i.e. any line whatever, satisfies the equation

$$\Sigma a \cdot AL = 0.$$

It is obvious, in the general case, that when all the points of the system, and all save one of the multiples are

given; by assigning a definite value to the last multiple, the position of the mean centre is determinate; and conversely *any point* whatever is the mean centre of a given system for multiples, all of which save two may be arbitrarily chosen.

## EXAMPLES.

1. The middle point of a right line is the mean centre of its extremities (Euc. I. 26).

2. The mean centre  $O$  of two points  $A$  and  $B$  for the multiples  $a$  and  $b$  divides the line  $AB$  inversely as the multiples, *i.e.*,

$$AO : BO = b : a.$$

The mean centre of the same points for the multiples  $a, -b$ , divides the line *externally* such that

$$AO : BO = b : a.$$

3. The mean centre  $O$  of a linear system of points  $A, B, C \dots$  for multiples each = 1 satisfies the equation  $\Sigma AO = 0$ .

4. The bisectors  $L, M, N$  of the sides of a triangle  $ABC$  are concurrent.

[For  $\Sigma AL = 0, \Sigma AM = 0$  and  $\Sigma AN = 0$ , hence each line passes through the mean centre (centroid or centre of gravity) of the vertices.]

5. The lines joining the middle points of the three pairs of opposite connectors  $BC$  and  $AD$ ,  $CA$  and  $BD$ ,  $AB$  and  $CD$  of four points  $A, B, C, D$  are concurrent, and each is bisected at the point of concurrence.\*

\* In the particular case when the fourth point  $D$  coincides with the orthocentre  $O$  of the triangle  $ABC$  we infer at once the well-known property :—

*The lines joining the middle points of the sides of a triangle with those of the segments towards the angles of the corresponding perpendiculars meet in a point and bisect each other.* From this it follows immediately (Euc. I. 4) that the six segments are equal, and that the circle passing through the middle points of the sides passes through the feet of the perpendiculars and bisects the segments of the latter towards the angles. This is the fundamental property of the *Nine-Points-Circle*.

6. The geometrical centre  $O$  of a regular polygon is the mean centre of the vertices  $A, B, C \dots$

[Join  $AO$  and  $BO$ . If the polygon be of an even order these lines ( $L$  and  $M$ ) will pass through the opposite vertices, and the perpendiculars from the remaining vertices are equal in pairs and opposite in sign; and if the polygon be of an odd order  $L$  and  $M$  bisect the opposite sides at right angles; therefore, etc.]

7.  $ABCD \dots$  is a regular cyclic polygon and  $L$  any line passing through its centre  $O$ ; prove that

$$AL + BL + CL + \dots = 0.$$

**52. Theorem.**—Any point  $O$  is the mean centre of the vertices of a triangle  $ABC$  for multiples proportional to the areas  $BOC, COA, AOB$ .

For letting  $L$  coincide with  $AOX$  and applying the relation  $\Sigma aAL = 0$  we have

$$b \cdot BL + c \cdot CL = 0,$$

or disregarding signs  $BL/CL = c/b$ .

Also since the triangles  $COA$  and  $AOB$  are upon the same base  $AO$ ,  $BL/CL = AOB/COA$ ; equating these values,

therefore 
$$\frac{b}{c} = \frac{COA}{AOB}.$$

Similarly 
$$\frac{c}{a} = \frac{AOB}{BOC}.$$

Hence 
$$a : b : c = BOC : COA : AOB.$$

If the point  $O$  is outside the triangle, and within the angle  $A$ , the multiples are proportional to

$$-BOC, COA \text{ and } AOB,$$

with similar results when  $O$  is within the angles  $B$  or  $C$ .

#### EXAMPLES.

1. The in-centre of a triangle is the mean centre of the vertices for multiples proportional to the sides.

2. The ex-centres are the mean centres for systems of multiples  $-a, b, c$ ;  $a, -b, c$ ;  $a, b, -c$ ; or quantities proportional to them.

3. If  $O, O_1, O_2, O_3$  denote the in- and ex-centres of a triangle, each is the mean centre of the remaining three for multiples,

$$s-a, s-b, s-c; s-b, s-c, -s, \text{ etc.}$$

[For the areas in the first case are  $O_2O_3O, O_3O_1O, O_1O_2O$ , and these are obviously proportional to  $s-a, s-b, s-c$ . Similarly for each of the ex-centres. Thus generally since  $-s:s-a:s-b:s-c = -1/r:1/r_1:1/r_2:1/r_3$ ; for the points  $O, O_1, O_2, O_3$  each is the mean centre of the remaining three for the corresponding multiples of the system  $-1/r, 1/r_1, 1/r_2, 1/r_3$ .]

4. Prove the following points are the mean centres of the vertices for the system of multiples written opposite to them.

Circum-centre	$\begin{cases} a \cos A, b \cos B, c \cos C, \\ \sin 2A, \sin 2B, \sin 2C. \end{cases}$
Orthocentre	$\tan A, \tan B, \tan C.$
Symmedian Point	$a^2, b^2, c^2.$
Brocard Points	$\frac{1}{b^2}, \frac{1}{c^2}, \frac{1}{a^2}; \frac{1}{c^2}, \frac{1}{a^2}, \frac{1}{b^2}.$

“Nine-Points” Centre  $a \cos(B-C), b \cos(C-A), c \cos(A-B).$ \*

5. The lines drawn from the vertices of a triangle to the points of contact of the in-circle are concurrent at the mean centre of the vertices for multiples  $r_1, r_2, r_3.$

6. The lines drawn to the internal points of contact of the three ex-circles meet at the mean centre of the vertices for multiples

$$1/r_1, 1/r_2, 1/r_3.$$

7. If a point  $O$  be the mean centre of the vertices for multiples  $l, m, n$ , its *Isotomic Conjugate* † is the mean centre for multiples the reciprocals of  $l, m, n$ .

7a. The *Isogonal Conjugate* † of  $O$  is the mean centre for multiples  $a^2/l, b^2/m, c^2/n$ .

\* From this it is evident that the sides of the triangle  $ABC$  meet the Nine-Points-Circle at angles  $B-C, C-A, A-B$ .

† Two points  $X$  and  $X'$  equidistant from the extremities of a line  $BC$  are called *Isotomic Conjugates* with respect to the line. It is easy to see, and it will be afterwards proved, that if the sides of a triangle  $ABC$  be divided isotomically in the pairs of points  $X, X'; Y, Y'; Z, Z'$ ; such that  $AX, BY$  and  $CZ$  are concurrent at a point  $O$ ; then  $AX',$

8. Any point  $O$  on the segment  $AB$  of the circum-circle of an equilateral triangle  $ABC$  is the mean centre of the vertices for multiples  $1/OA, 1/OB, -1/OC$ .

9. The mean centre of  $O, O_1, O_2, O_3$  is in Ex. 3 the circum-centre of the triangle.

10. The centre of Taylor's circle is the mean centre of the vertices of the pedal triangle of  $ABC$  for multiples

$$a \cos(B-C), b \cos(C-A), c \cos(A-B).$$

11. The mean centre  $O$  of the vertices of  $ABC$  for multiples  $l, m, n$  is the mean centre of the vertices of the pedal triangle  $PQR$  of  $O$  for multiples  $a^2/l, b^2/m, c^2/n$ .

[From the figure of Art. 23, Ex. 1, we have

$$QOR : ROP : POQ = OQ \cdot OR \sin A : OR \cdot OP \sin B : OP \cdot OQ \sin C \\ = a/OP : b/OQ : c/OR \dots \dots \dots (1)$$

$$\text{But } OP : OQ : OR = BOC/a : COA/b : AOB/c \\ = l/a : m/b : n/c.$$

Substituting these values in (1); therefore, etc.]

12. The symmedian point  $O$  of any triangle is the centroid of the pedal triangle of  $O$ .

[For  $BOC : COA : AOB = a^2 : b^2 : c^2$  by Art. 16, Ex. 2 (2).]

13. The lines joining  $A, B, C$  to the corresponding vertices of Brocard's first triangle are concurrent, and the point of concurrence is the mean centre of the vertices of  $ABC$  for multiples the reciprocals of  $a^2, b^2, c^2$ .

[For it has been shown that it is the isotomic conjugate of the symmedian point, Art. 30, Ex. 3.]

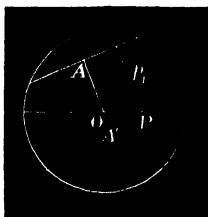
14. If perpendiculars be let fall from any point  $P$  on the sides of a regular polygon; the mean centre of their feet lies on the line joining  $P$  to the circum-centre.

$BY', CZ'$  are also concurrent at  $O'$ . The points  $O$  and  $O'$  are termed *Isotomic Conjugates with respect to the triangle  $ABC$* .

If the pairs of lines  $AX, AX'$ , etc., are equally inclined to the sides  $b$  and  $c$ , etc., they are *Isogonal Conjugates* with respect to the angles; and if  $AX, BY, CZ$  are concurrent,  $AX', BY', CZ'$  are also concurrent. The points of concurrence are *Isogonal Conjugates with respect to the triangle*.



[Through  $O$  draw  $OA'$  parallel and  $PA'$  perpendicular to  $p_1$ . The projection of  $p_1$  on  $OP$  = projection of  $AA'$ ; but  $A, B, C, \dots$  and

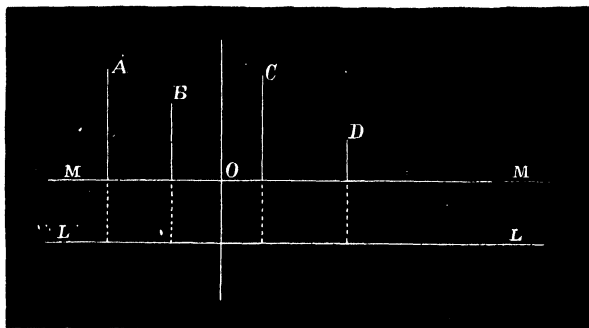


$A', B', C', \dots$  are the vertices of regular polygons, whose mean centres are both on  $OP$ . Therefore the sum of the projections of  $p_1 \dots$  on  $OP=0$ .]

**53. Theorem.**—For any line  $L$  to prove that

$$\Sigma a \cdot AL = \Sigma(a)OL.$$

Draw  $M$  through  $O$  parallel to  $L$ .



Then  $AL = AM + OL,$

$$BL = BM + OL,$$

$$CL = CM + OL, \text{ etc.}$$

Multiplying these equations respectively by  $a, b, c, \dots$  and adding, we have

$$\Sigma(a \cdot AL) = \Sigma(a \cdot AM) + \Sigma(a)OL;$$

but  $\Sigma(a \cdot AM) = 0$  since  $M$  passes through the mean centre; therefore, etc.

This property enables us to find the mean centre. For by taking a line  $L$  in an arbitrary position and calculating  $\Sigma(a \cdot AL)/\Sigma(a)$  we have for the locus of  $O$  a line parallel to  $L$  at this distance from it. Again, take a line in another position and construct the locus of  $O$  as before. The intersection of these loci is the point required.

COR. 1. If  $\Sigma a \cdot AL$  is a constant, the line  $L$  touches, or envelopes, a circle concentric with  $O$ .

COR. 2. If the multiples are all equal  $\Sigma AL = n \cdot OL$ , where  $n$  denotes the number of points in the system.

COR. 3. For systems of points and multiples and their mean centres

$$\begin{aligned} A_1 B_1 C_1 \dots, a_1 b_1 c_1 \dots, O_1, \\ A_2 B_2 C_2 \dots, a_2 b_2 c_2 \dots, O_2, \\ \vdots \\ A_n B_n C_n \dots, a_n b_n c_n \dots, O_n, \end{aligned}$$

the mean centre  $O$  of all the points and their corresponding multiples is the mean centre of  $O_1, O_2, \dots, O_n$  for the multiples  $\Sigma(a_1), \Sigma(a_2), \dots, \Sigma(a_n)$ .

[For since  $\Sigma a_1 A_1 L = \Sigma(a_1) O_1 L$ ,  $\Sigma a_2 A_2 L = \Sigma(a_2) O_2 L$ , etc., on adding these equations

$$\begin{aligned} \Sigma a_1 A_1 L + \Sigma a_2 A_2 L + \dots + \Sigma a_n A_n L &= \Sigma(a_1) O_1 L + \dots \\ &= \Sigma(\Sigma a_i) O L.] \end{aligned}$$

Hence the mean centre of a system of points can be found as follows:—Find the mean centre  $O_1$  of two of the points  $A$  and  $B$ ; next find the mean centre of  $O_1$  and  $C$  for multiples  $a + b, c$ . Denote this by  $O_2$ , and find the mean centre of  $O_2$  and  $D$  for multiples  $a + b + c, d$ , and so on. When the entire system has thus been exhausted the last mean centre found is that of the system.

## EXAMPLES.

1. The sum of the distances of the vertices of a triangle from any line is equal to three times the distance of its centroid from the line.

2. Draw a tangent to a circle such that  $\Sigma a \cdot AL$  may be a maximum, minimum, or have any given value.

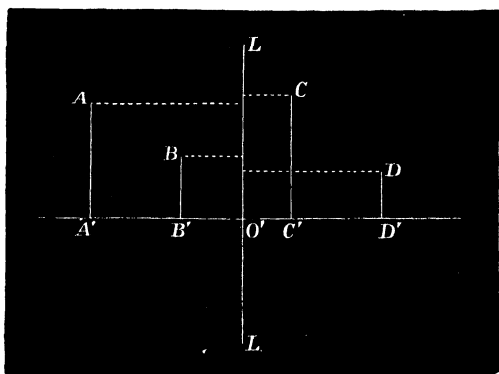
[The extremities of the diameter passing through the mean centre are obviously the points of contact in the extreme cases. The general case reduces to draw a common tangent to two circles.]

3. If  $L$  touches the in-circle  $\Sigma a \cdot AL = 2\Delta$  when the multiples are equal to the sides of the triangle.

3a. For the ex-circle to the side  $c$  the equation becomes

$$aAL + bBL - cCL = 2\Delta.$$

4. The projection of the mean centre on any line is the mean centre of the projections of the system of points on the line.



[Let the projections be denoted by  $O', A', B', C' \dots$  and  $L$  the line  $OO'$ . Then  $A'O' = AL$ ,  $B'O' = BL$ , etc. Hence

$$\Sigma a \cdot A'O' = \Sigma a \cdot AL = 0; \text{ therefore, etc.}]$$

5. If  $O, O_1, O_2, O_3$  denote the in- and ex-centres of a triangle,  
 $(s-a)O_1L + (s-b)O_2L + (s-c)O_3L = s \cdot OL$ .\*

---

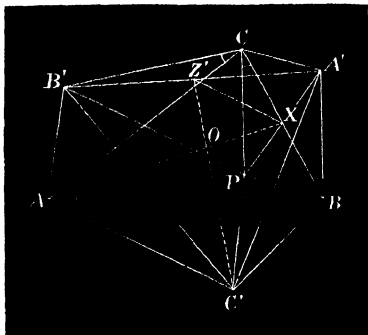
\* This relation may be otherwise written :—

$$\frac{O_1L}{r_1} + \frac{O_2L}{r_2} + \frac{O_3L}{r_3} = \frac{OL}{r}.$$

[For  $O$  is the mean centre of the remaining points for multiples  $s-a, s-b, s-c$  (Art. 52, Ex. 3), and since

$$\Sigma(s-a)=s; \text{ therefore, etc.}]$$

6. Let three similar triangles  $BCA', CAB'$  and  $ABC'$  be described on the sides of  $ABC$  in the same aspect; to prove that the mean centres of the triangles  $ABC$  and  $A'B'C'$  coincide (Brocard).



[Let  $X$  be the middle point of  $BC$  and  $Z'$  of  $A'B'$ . Complete the parallelogram  $BA'CP$ . Join  $AX, C'Z', Z'X$  and  $PB'$ . The triangles  $BPC$  and  $B'CA$  are similar, therefore  $CP/CB = B'C/AC$  (Euc. VI. 4), or by alternation  $B'C/CP = AC/BC$ ; also the angles  $B'CP$  and  $ACB$  are equal, therefore the triangles  $B'PC$  and  $ABC$  are similar (Euc. VI. 6); hence  $CB'/BP = CA/AB$ ; alternately  $CB'/CA = PB'/AB$ ; but  $CB'/CA = C'A/AB$  (hyp.); therefore  $PB'/AB = C'A/AB$  from which

$$PB' = AC'.$$

Again  $\angle PB'C = \angle BAC'$ , to these add the equals  $ACB'$  and  $BAC''$  respectively; therefore  $PB'$  and  $AC'$  are parallel. But  $Z'X$  is parallel and equal to half of  $PB'$ ; therefore it is parallel and equal to half of  $AC'$ . Hence the medians  $AX$  and  $C'Z'$  trisect each other.\* Otherwise thus :†—Let another triangle  $ABC''$  be described below the base  $AB$  symmetrically equal to  $ABC'$ . It is easy to see that

\* For another proof see Milne's *Companion to the Weekly Problem Papers*, Art. 123.

† *Educational Times*. Reprint. Vol. liv., p. 102.

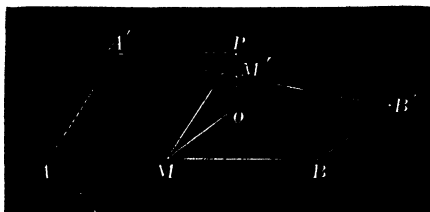
the triangles  $ABA'$  and  $CBC''$  are equal in area; similarly  $ABB'$  and  $CAC''$  are equal. By addition we have  $ABA' + ABB' = ABC + ABC''$  or  $ABA' + ABB' - ABC = ABC$ , i.e. the algebraic sum of the perpendiculars on  $AB$  from  $A'$ ,  $B'$ ,  $C'$  = the perpendicular from  $C$  on  $AB$ . Similar results are obtained for the sides  $BC$  and  $CA$ ; therefore, etc. Syamadas Mukhopadhyay.]

7. If two points  $A$  and  $B$  be displaced to new positions  $A'$  and  $B'$ , their mean centre  $M$  for any multiples is displaced to  $M'$  found by the following construction:—

Through  $M$  draw lines  $MP$  and  $MQ$  equal and parallel to  $AA'$  and  $BB'$  respectively. Join  $PQ$  and divide it in  $M'$  such that  $PM'/QM' = AM/BM$ .

[For since  $AA'PM$  and  $BB'QM$  are parallelograms,  $A'P = AM$  and  $B'Q = BM$ ; therefore by similar triangles  $PA'M'$  and  $QB'M'$ ,

$$\frac{A'P}{B'Q} = \frac{A'M'}{B'M'} = \frac{PM}{QM'} : \text{therefore, etc.}]$$



8. If three points  $A$ ,  $B$  and  $C$  be displaced to new positions  $A'$ ,  $B'$  and  $C'$ , their mean centre  $M$  is displaced to  $M'$  found by the following construction:—

Through  $M$  draw lines  $MP$ ,  $MQ$  and  $MR$  equal and parallel to the displacements  $AA'$ ,  $BB'$  and  $CC'$  respectively;  $M'$  is the mean centre of  $P$ ,  $Q$ ,  $R$ .

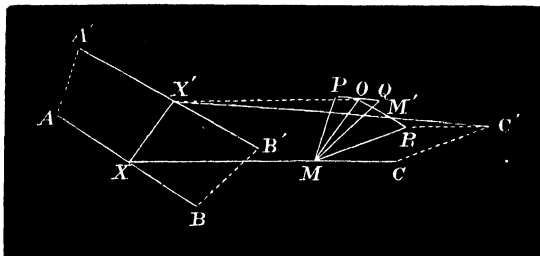
[For let  $X$  denote the mean centre of  $A$  and  $B$ ,  $X'$  which is found by Ex. 7 of  $A'$  and  $B'$ . Draw  $MO$  equal and parallel to  $XX'$ . Join  $OX'$ ,  $RC'$  and  $X'C'$ .

It is evident by parallels that  $O$  is the mean centre of  $P$  and  $Q$ ; also  $MX = OX'$  and  $MC = RC'$ ; therefore in the similar triangles  $OM'X'$  and  $RM'C'$ ,

$$\frac{M'X'}{M'C'} = \frac{OX'}{RC'} = \frac{MX}{MC} \dots \dots \dots (1)$$

hence  $M'$  is the mean centre of  $X'$  and  $C'$ , that is of  $A'$ ,  $B'$  and  $C'$ , for the same multiples that  $M$  is of  $A$ ,  $B$  and  $C$ .

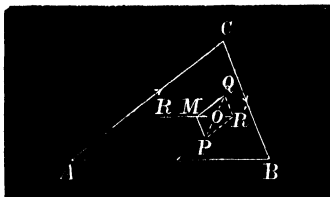
But each of the ratios in (1) is equal to  $M'O/M'R$ ; therefore  $M'$  is the mean centre of  $O$  and  $R$ , that is of  $P$ ,  $Q$  and  $R$  for the same set of multiples.



NOTE.—The construction for the displaced mean centre may in the same manner be extended to the quadrilateral and generally to a polygon of any number of sides.

Hence for two systems of points  $A, B, C, \dots$  and  $A', B', C', \dots$  and their mean centres  $M$  and  $M'$  for the same set of multiples  $a, b, c \dots$  if we draw through  $M$  parallels  $MP, MQ, MR, \dots$  equal to  $AA', BB', CC', \dots$  respectively, the mean centre of the third system  $P, Q, R, \dots$  for the same multiples coincides with  $M'$ .

9. If through any point  $M$  are drawn  $MP, MQ$  and  $MR$  parallel and proportional to the sides of a triangle  $ABC$ , the mean centre of  $P, Q$  and  $R$  for multiples each equal to unity coincides with  $M$ .



[By Ex. 8, or thus :—Complete the parallelograms  $PMQR'$  and draw  $MR'$ .

Since  $\frac{PM}{PR} = \frac{QM}{QR} = \frac{a}{b}$  and the angles at  $P$  and  $Q$  equal, the triangles  $PMR'$  and  $QMR'$  are similar, hence  $MR = MR' = 2MO$ , and  $O$

is the mean centre of  $P$  and  $Q$ , and therefore  $M$  is the mean centre of  $P, Q, R$ .]

10. Prove the similar property for the quadrilateral; and generally :—

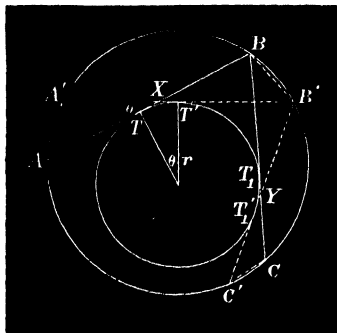
If through any point  $M$  lines are drawn parallel and proportional to the sides of a polygon; the mean centre of their extremities for multiples each = 1 coincides with  $M$ .

11. If a system of points  $A, B, C, \dots$  be displaced to  $A', B', C', \dots$  such that  $AA', BB', CC', \dots$  are parallel and proportional to the sides of a polygon, the mean centre of the system remains a fixed point.

[By aid of Exs. 8 and 10.]

12. **Weill's Theorem.**—A variable polygon is inscribed to one circle and escribed to another; to prove that the mean centre of the points of contact of its sides with the latter circle is a fixed point.

[Let  $ABC \dots$  denote the polygon,  $A'B'C' \dots$  a consecutive position,  $T$  and  $T'$  the points of contact of  $AB$  and  $A'B'$  with the circle of radius  $r$ ;  $\theta$  the small angle between  $AB$  and  $A'B'$ , and  $X$  their intersection.]



The triangles  $AA'X$  and  $BB'X$  are similar, hence  $BB'/AA' = BX/A'X$  and

$$\frac{BB'}{AA' + BB'} = \frac{BX}{BX + A'X} = \frac{BX}{BX + AX} \text{ in the limit} = \frac{BX}{AB} \dots\dots(1)$$

Similarly

$$\frac{BB'}{BB' + CC'} = \frac{BY}{BC} \dots\dots\dots(2)$$

Also, since  $AB$  and  $A'B'$  are indefinitely near to one another,  $X$  is indefinitely near to the point of contact  $T$ , and  $BX$  and  $BY$  are therefore equal because they are tangents from the same point to a circle.

Dividing (2) by (1)

$$\frac{AB}{BC} = \frac{AA' + BB'}{BB' + CC'} \dots \dots \dots (3)$$

Again  $\frac{AA'}{AX} = \frac{BB'}{BX} = \frac{\theta}{\sin A'}$  (Rule of Sines),

hence  $\frac{\theta}{\sin A'} = \frac{AA' + BB'}{AB} ;$

but  $AB = (\text{diameter of } ABC) \times \sin A'$

and  $TT' = 2r\theta ;$

therefore  $TT' \propto AA' + BB' \propto AB$  (by 3).

Thus as the polygon  $ABC\dots$  varies, its points of contact are displaced for each consecutive position in the direction of its sides, and proportional to them; therefore the mean centre is a fixed point.

NOTE.—If the side  $BC$  is a variable tangent to a third circle of radius  $r$ , the result of dividing (2) by (1) is

$$\frac{AB}{BC} = \frac{AA' + BB'}{BB' + CC'} \cdot \frac{BX}{BY} ;$$

therefore if the three circles are so related that  $BX/BY$  is a constant ratio  $k$ ,

$$\frac{AB}{BC} = k \cdot \frac{AA' + BB'}{BB' + CC'}$$

and  $TT'/T_1T_1' = r/kr' \cdot AB/BC.]$

13. The mean centres of the vertices of any polygon and of similar triangles similarly described on its sides coincide (M'Cay).

[Let the vertices of the triangles on the sides  $AB, BC, CD \dots$  be  $A', B, C' \dots$  respectively.

Since  $AA' : BB' : CC' \dots = AB : BC : CD \dots$  and are inclined to the sides of the polygon at the same angle; we may regard the vertices of the given polygon displaced to  $A'B'C' \dots$  distances proportional and parallel to its sides turned through that angle (cf. Ex. 6).]\*

---

\* The proofs of Examples 11-13 were communicated to the Author by Mr. Charles M'Vicker.



14. Through the centre  $O$  of a regular polygon any line is drawn meeting the sides in  $A', B', C', \dots$  to prove that  $\sum \frac{1}{OA'} = 0$ .

[Let  $M$  be the middle point of one side, then  $MA'O$  is a right-angled triangle, and if a perpendicular  $MM'$  be let fall on the hypotenuse we have

$$OA' \cdot OM' = r^2 \text{ or } \sum \frac{1}{OA'} = \frac{1}{r^2} \sum OM' = 0. \quad \text{Art. 50. See Art. 3, Ex. 9.}]$$

54. **Theorem.**—*For any system of points  $A, B, C, \dots$  their mean centre  $O$ , and any line  $L$ ; to prove that*

$$\sum a \cdot AL^2 = \sum a \cdot AL'^2 + \sum (a) OL^2,$$

where  $L'$  is the line through  $O$  parallel to  $L$ .

$$\text{For } AL = AL' + OL; \therefore AL^2 = AL'^2 + OL^2 + 2AL' \cdot OL;$$

$$BL = BL' + OL; \therefore BL^2 = BL'^2 + OL^2 + 2BL' \cdot OL;$$

.....

Multiplying these equations by  $a, b, c, \dots$  respectively and adding results,

$$\sum a \cdot AL^2 = \sum a \cdot AL'^2 + \sum (a) OL^2 + 2OL \sum (a \cdot AL'),$$

but  $\sum a \cdot AL' = 0$  (Art. 50); therefore, etc.

COR. 1. When the multiples are equal

$$\sum AL^2 = \sum AL'^2 + nOL^2,$$

also since  $\sum AL = n \cdot OL$ ;  $OL$  is the arithmetical mean of the several lines  $AL, BL, CL \dots$ , and  $AL', BL' \dots$  the several differences between each and their mean.

Hence, *the sum of the squares of  $n$  quantities =  $n$  times the square of their mean value + the sum of squares of the  $n$  differences*; or if the quantities are the segments of a line this property may be stated: *the sum of the squares of the unequal parts = the sum of the squares of the equal parts + the sum of the squares of the  $n$  differences*. This property is obviously an extension of Euc. II. 9, 10.

COR. 2. For any two parallel lines  $L$  and  $M$ ,

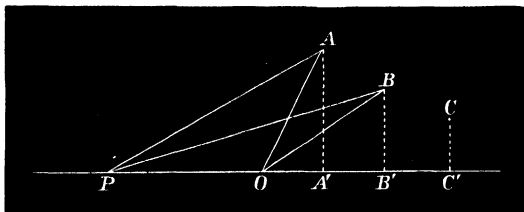
$$\sum a \cdot AL^2 - \sum a \cdot AM^2 = \sum (a)(OL^2 - OM^2).$$

**55. Theorem.**—For any point  $P$  to prove that

$$\Sigma a \cdot AP^2 = \Sigma a \cdot AO^2 + \Sigma(a)OP^2.$$

Project the system of points on the line  $OP$  and denote their projections by  $A', B', C', \dots$

Then (Euc. III. 12-13),



$$AP^2 = AO^2 + OP^2 + 2OP \cdot OA'.$$

Similarly  $BP^2 = BO^2 + OP^2 + 2OP \cdot OB'$  etc.

Multiplying these equations by  $a, b, c \dots$  and adding the results,

$$\Sigma a \cdot AP^2 = \Sigma a \cdot AO^2 + \Sigma(a)OP^2 + 2OP \Sigma a \cdot OA',$$

but  $O$  is the mean centre of the system  $A', B', C' \dots$  (Art. 53, Ex. 4); therefore  $\Sigma a \cdot OA' = 0$ .

**COR. 1.** If the  $n$  multiples are equal

$$\Sigma AP^2 = \Sigma AO^2 + n \cdot OP^2.$$

**COR. 2.** For a regular cyclic polygon the sum of the squares of the distances of any point on the circle from the  $n$  vertices is constant and  $= 2nR^2$ .

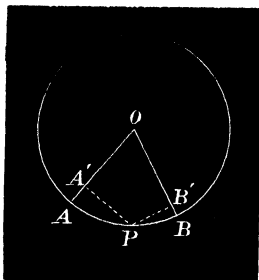
**COR. 3.** If  $\Sigma a \cdot AP^2$  is constant, the locus of  $P$  is a circle concentric with  $O$  the square of whose radius is

equal to 
$$\frac{\Sigma a \cdot AP^2 - \Sigma a \cdot AO^2}{\Sigma(a)}.$$

**COR. 4.**  $\Sigma a \cdot AP^2$  is a minimum when  $P$  coincides with  $O$ . See Art. 16, Ex. 3.

## EXAMPLES.

1.  $ABCD \dots$  is a regular cyclic polygon,  $O$  the centre,  $R$  the radius, and  $P$  any point on the circle to prove that the sum of the squares of the perpendiculars from  $P$  on the radii  $OA, OB, OC \dots$   $= \frac{1}{2}nR^2$ .



[Denote the feet of the perpendiculars by  $A', B', C' \dots$ . The circle on  $OP$  as diameter passes through these points (Euc. III. 31); also since  $A'B', B'C', \dots$  subtend equal angles ( $2\pi/n$ ) at  $O$ , a point on the circle,  $A'B'C' \dots$  is a *regular cyclic polygon*. Hence (Cor. 2)

$$\Sigma PA'^2 = 2n(\frac{1}{2}OP)^2 = \frac{1}{2}nR^2.$$

Similarly

$$\Sigma OA'^2 = \frac{1}{2}nR^2.]$$

2. For any line  $L$  passing through  $O$ ,  $\Sigma AL^2 = \frac{1}{2}nR^2$ .

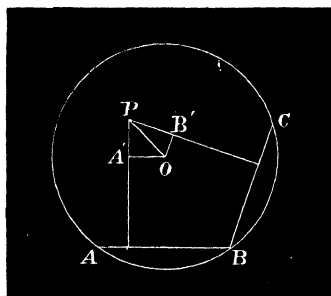
[Let  $L$  coincide with  $OP$ . By similar triangles

$$AL = PA', BL = PB', \text{ etc.}$$

therefore

$$\Sigma PA'^2 = \Sigma AL^2 = \frac{1}{2}nR^2 \text{ by Ex. 1.}]$$

3. The sum of the squares of the perpendiculars  $p_1, p_2, p_3 \dots p_n$



from *any* point  $P$  upon the sides of the polygon is equal to  $n(r^2 + \frac{1}{2}\delta^2)$ , where  $r$  is the radius of the in-circle and  $\delta = OP$ .

[Through  $O$  draw parallels  $OA', OB', OC' \dots$  to the sides of the polygon meeting the corresponding perpendiculars from  $P$  in  $A', B', C', \dots$  As before  $A'B'C' \dots$  is a regular cyclic polygon inscribed in the circle on  $OP$  as diameter.

Since the sum of the perpendiculars is constant and  $=nr$

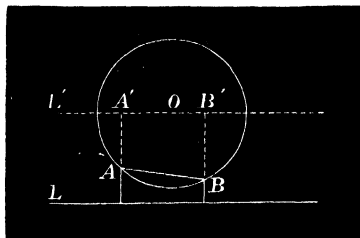
$$\Sigma p_1^2 = nr^2 + \Sigma PA'^2 \dots\dots\dots (\text{Art. 54}) (1)$$

but  $\Sigma PA'^2 = \frac{1}{2}n\delta^2$  (Ex. 1),

substituting this value in (1); therefore, etc.]

4. In Ex. 3 if  $P$  is on the in-circle  $\Sigma p_1^2 = \frac{3}{2}nr^2$ .

5. If  $\pi_1, \pi_2, \pi_3, \dots$  denote the distances of the vertices from any line  $L$  and  $\delta = OL$ ,  $\Sigma \pi_1^2 = n(\delta^2 + \frac{1}{2}R^2)$ .



[Through  $O$  draw  $L'$  parallel to  $L$  and let  $A', B', C' \dots$  be its intersections with  $AL, BL, CL \dots$  respectively.

Since  $\Sigma AL = nOL$  (Art. 53),

$$\Sigma AL^2 = n \cdot OL^2 + \Sigma AA'^2 \quad (\text{Art. 54}),$$

but  $\Sigma AA'^2 = \frac{1}{2}nR^2$  (Ex. 2);

therefore by substitution  $\Sigma AL^2 = n(OL^2 + \frac{1}{2}R^2)$ ,

or  $\Sigma \pi_1^2 = n(\delta^2 + \frac{1}{2}R^2)$ .

5a. If  $L$  is a tangent to the circum-circle

$$\Sigma \pi_1^2 = \frac{3}{2}nR^2.$$

6. If  $P$  be a point on the circum-circle of a regular polygon  $ABC \dots$ ,  $\Sigma PA^4 = 6nR^4$ .

[Draw  $OP$  and produce it to meet the circle again in  $Q$ , and let  $A', B', C' \dots$  be the projections of the vertices on this line. Since  $PAQ$  is a right-angled triangle,

$$PQ \cdot PA' = PA^2.$$

Squaring, we have  
therefore

$$4R^2 \cdot PA'^2 = PA^4,$$

$$4R^2 \Sigma PA'^2 = \Sigma PA^4.$$

But

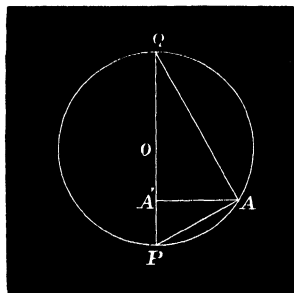
$$\Sigma PA'^2 = nR^2 + \Sigma OA'^2, \quad (\text{Art. 54, Cor. 1.})$$

and

$$\Sigma OA'^2 = \frac{1}{2}nR^2. \quad (\text{Ex. 2.})$$

Substituting

$$\Sigma PA^4 = 4R^2(nR^2 + \frac{1}{2}nR^2) = 6nR^4.]$$



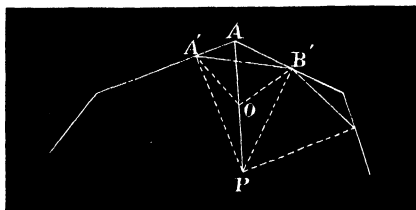
7. If  $a, b, c$  denote the sides of a triangle  $ABC$  and  $P$  any point on the in-circle,  $\Sigma a \cdot AP^2 = \Sigma a \cdot AO^2 + 2r\Delta$ .

8. If  $ABC$  be an equilateral triangle, and  $L$  a tangent to the in-circle,

$$\frac{1}{AL} + \frac{1}{BL} + \frac{1}{CL} = 0.$$

[For  $AL + BL + CL = 3r$ , therefore on squaring  $\Sigma AL^2 + 2\Sigma BL \cdot CL = 9r^2$ . Also  $\Sigma AL^2 = 3r^2 + \frac{3}{2}R^2$ , or since  $R = 2r$ ,  $\Sigma AL^2 = 9r^2$ ; hence  $\Sigma BL \cdot CL = 0$ , therefore, etc.]

9. Perpendiculars are let fall from  $P$  on the sides of any polygon  $ABC \dots$  and their feet joined; prove that if the area of the in-



scribed figure  $A'B'C' \dots$  is constant, the locus of  $P$  is a circle concentric with the mean centre of  $A, B, C, \dots$  for the multiples in  $2A, \sin 2B, \sin 2C, \dots$

[Let  $O$  be the middle point of  $AP$ . Then

$$2A'OB' = 2A'B'P - AA'B'P;$$

hence

$$2\Sigma A'OB' = 2\Sigma PA'B' - \Sigma PAA'B',$$

or

$$\frac{1}{2} \cdot PA^2 \sin 2A = 2A'B'C' \dots - ABC \dots$$

Therefore  $\Sigma \sin 2A \cdot AP$  is constant ....

For a triangle the mean centre of  $A, B, C$  for multiples  $\sin 2A, \sin 2B, \sin 2C$  is the circum-centre, showing Art. 23, Ex. 2, to be a particular case of this theorem. M'Vicker.]

**56. Theorem.**—If  $\Sigma AB$  denote the sum of the mutual distances of a system of points  $A, B, C \dots$  from each other; to prove that  $\Sigma(ab \cdot AB)^2 = \Sigma(a) \cdot \Sigma(a \cdot AO^2)$ .

In Art. 55 if we suppose  $P$  to coincide with each point of the system successively we have the following relations :—

$$a \cdot AA^2 + b \cdot AB^2 + c \cdot AC^2 + \dots = \Sigma a \cdot AO^2 + \Sigma(a)OA^2,$$

$$a \cdot BA^2 + b \cdot BB^2 + c \cdot BC^2 + \dots = \Sigma a \cdot AO^2 + \Sigma(a)OB^2,$$

$$a \cdot CA^2 + b \cdot CB^2 + c \cdot CC^2 + \dots = \Sigma a \cdot AO^2 + \Sigma(a)OC^2,$$

.....

Multiplying these results by  $a, b, c \dots$  respectively and adding  $2\Sigma ab \cdot AB^2 = \Sigma(a) \cdot \Sigma a \cdot AO^2 + \Sigma(a) \cdot \Sigma a \cdot AO^2$ , therefore  $\Sigma ab \cdot AB^2 = \Sigma(a) \cdot \Sigma a \cdot AO^2$ .

COR. 1. If the multiples are each equal to unity,

$$\Sigma AB^2 = n \cdot \Sigma AO^2.$$

COR. 2. The sum of the squares of all the lines joining the vertices of a regular polygon  $= n^2 R^2$ ; where  $R$  is the radius of the circum-circle.

COR. 3. For three points  $A, B, C$ , the sum of the squares of the sides of a triangle  $=$  three times the sum of the squares of the lines joining the vertices to the centroid; or three times the sum of the squares of the sides  $=$  four times the sum of the squares of the medians.

COR. 4. If  $O$  be the in-centre and  $a, b, c$  the sides of a triangle  $ABC$   $\Sigma(ab \cdot AB^2) = \Sigma(a) \cdot \Sigma a \cdot AO^2$  reduces to (Art. 52, Ex. 1)

$$abc(a+b+c) = (a+b+c)\Sigma a \cdot AO^2, \\ \Sigma a \cdot AO^2 = abc,$$

with analogous results for  $O_1, O_2$ , and  $O_3$ .

COR. 5. The sum of the squares of the six lines joining the centres of the in- and ex-circles  $= 48R^2$ .

Since the centre  $O$  of the circum-circle is (Art. 52, Ex. 9.) the mean centre of  $O_1, O_2, O_3, O_4$ ,

$$\Sigma O_1 O_2 = 4 \Sigma O O_1^2 = 4 \{ R^2 - 2Rr + \Sigma(R^2 + 2Rr_1) \} \\ = 16R^2 + 8R(r_1 + r_2 + r_3 - r),$$

but  $r_1 + r_2 + r_3 - r = 4R$ ; therefore, etc.\*

#### EXAMPLES.

1. If  $S$  denote the symmedian point of a triangle,

$$a^2 AS^2 + b^2 BS^2 + c^2 CS^2 = \frac{3a^2 b^2 c^2}{a^2 + b^2 + c^2} \quad (\text{Art. 52, Ex. 4.})$$

2. For the Brocard points  $\Omega, \Omega'$ ,

$$\alpha^\circ. \quad \frac{A\Omega^2}{b^2} + \frac{B\Omega^2}{c^2} + \frac{C\Omega^2}{a^2} = 1.$$

$$\beta^\circ. \quad \frac{A\Omega'^2}{c^2} + \frac{B\Omega'^2}{a^2} + \frac{C\Omega'^2}{b^2} = 1. \quad (\text{Art. 52, Ex. 4.})$$

3. The distance  $OP$  of any point  $P$  from the in-centre of a triangle is given by the equation

$$\Sigma a \cdot AP^2 = abc + \Sigma(a) \cdot OP^2.$$

[Eliminating  $\Sigma a \cdot AO^2$  between the equations,

$$\Sigma a \cdot AP^2 = \Sigma a \cdot AO^2 + \Sigma(a) \cdot OP^2,$$

and

$$\Sigma(ab \cdot AB^2) = \Sigma(a) \cdot \Sigma a \cdot AO^2,$$

the above result follows.]

\* Otherwise thus :—Since  $O_1$  is the orthocentre of  $O_2 O_3 O_4$ , if perpendiculars  $OX, OY, OZ$  be drawn to the sides from the circum-centre  $O$  of  $O_2 O_3 O_4$ ,  $O_1 O_2 = 2OX$ ,  $O_1 O_3 = 2OY$ , ... ; also  $OO_2 = 2R$ ; hence

$$O_1 O_4^2 + O_2 O_3^2 = 4(2R)^2 = 16R^2,$$

therefore

$$\Sigma O_2 O_3^2 = 48R^2.$$

4. If  $P$  coincides with the circum-centre, prove the following where  $D, D_1, D_2, D_3$  are the distances of the circum-centre from the in- and ex-centres :—

$$D^2 = R^2 - 2Rr; \quad D_1^2 = R^2 + 2Rr_1, \text{ etc., etc.}$$

5. Prove that the distance  $\delta$  of the symmedian point  $S$  from the circum-centre  $O$  of a triangle  $ABC$  is given by the equation

$$\delta^2 = R^2 - \frac{3a^2b^2c^2}{(a^2 + b^2 + c^2)^2}$$

[For any point  $P$  (Art. 52, Ex. 4)  $\Sigma a^2 AP^2 = \Sigma a^2 AS^2 + \Sigma (a^2) \delta^2$ , letting  $P$  coincide with  $O$ ; therefore

$$(a^2 + b^2 + c^2)R^2 = \frac{3a^2b^2c^2}{a^2 + b^2 + c^2} + (a^2 + b^2 + c^2)\delta^2;$$

hence \*

$$\delta^2 = R^2 - \frac{3a^2b^2c^2}{(a^2 + b^2 + c^2)^2};$$

therefore, etc.]

6. The distances of  $\Omega$  and  $\Omega'$  from the circum-centre are given by the equations  $O\Omega = O\Omega' = R \sqrt{1 - 4 \sin^2 \omega}$ .

$$\left[ \text{For } \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{4R^2} \Sigma \operatorname{cosec}^2 A = \frac{1}{4R^2} \operatorname{cosec}^2 \omega. \right]$$

7. For the in-centre  $O_1$  and the ex-centres  $O_2, O_3, O_4$  prove the relations

$$\begin{aligned} \alpha^\circ. \quad & \frac{O_1 O_2^2}{r_1} + \frac{O_1 O_3^2}{r_2} + \frac{O_1 O_4^2}{r_3} = 8R. \\ \beta^\circ. \quad & \frac{O_3 O_4^2}{r_2 r_3} + \frac{O_4 O_2^2}{r_3 r_1} + \frac{O_2 O_3^2}{r_1 r_2} = \frac{8R}{r}. \end{aligned}$$

8. For any point  $P$

$$(s-a)PO_1^2 + (s-b)PO_2^2 + (s-c)PO_3^2 - sPO^2 = 2abc.$$

9. Find the following expression for the square of the distance  $\delta$  between the circum- and ortho-centre of a triangle  $ABC$ .

$$\begin{aligned} \delta^2 &= R^2(1 - 8 \cos A \cos B \cos C) \\ &= \Sigma a^2(a^2 - b^2)(a^2 - c^2)/16\Delta^2. \end{aligned}$$

[By the previous method, or more simply by finding the area of the pedal triangle of  $ABC$ , ( $2 \text{ area} = R^2 \sin 2A \sin 2B \sin 2C$ ), and using Art. 23, Ex. 1, and reducing.]

\* This expression is equivalent to

$$\delta^2 = R^2 \sec^2 \omega (1 - 4 \sin^2 \omega),$$

where  $\omega$  is the Brocard angle.



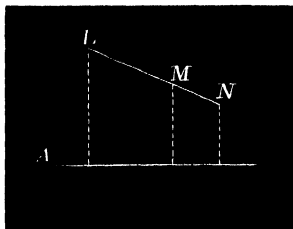
## RECIPROCAL THEOREMS.

57. **Theorem.**—For any two points  $M$  and  $N$ , and systems of lines  $A, B, C, \dots$  and multiples  $a, b, c, \dots$  having given  $\Sigma a \cdot MA = 0$  and  $\Sigma a \cdot NA = 0$  to prove that

$$\Sigma a \cdot LA = 0,$$

where  $L$  is any point on the line  $O$  connecting  $M$  and  $N$ .

For  $MN \cdot LA + NL \cdot MA + LM \cdot NA = 0.$



Similarly for the lines  $B$  and  $C$ ,

$$MN \cdot LB + NL \cdot MB + LM \cdot NB = 0,$$

$$MN \cdot LC + NL \cdot MC + LM \cdot NC = 0.$$

Multiplying these equations respectively by  $a, b, c, \dots$  and adding, we get

$$MN \Sigma a \cdot LA + NL \Sigma a \cdot MA + LM \Sigma a \cdot NA = 0 \dots (1)$$

hence if  $\Sigma a \cdot MA$  and  $\Sigma a \cdot NA$  each  $= 0$ ,  $\Sigma a \cdot LA = 0$ , for any other point  $L$  on the line  $MN$ .

More generally: If  $\Sigma a \cdot MA$  and  $\Sigma a \cdot NA$  are equal  $\Sigma a \cdot LA$  has the same value.

For, let  $\Sigma a \cdot MA = \Sigma a \cdot NA = k$ ; substituting in (1)

$$MN \Sigma a \cdot LA + (NL + LM)k = 0;$$

dividing by  $MN (= LN + ML)$  and transposing

$$\Sigma a \cdot LA = k.$$

Hence, the locus of a point  $L$  such that the sum of given multiples of the perpendiculars from it upon a system of lines  $A, B, C, \dots$  is constant ( $\Sigma a \cdot LA = k$ ) is a right line.

**Def.** When the constant vanishes  $\Sigma a \cdot LA = 0$ , the locus  $O$  is termed the *Central Axis* of the system of lines for the given system of multiples.

It is evident that the central axis is one of a system of parallel lines obtained by taking different values,  $k_1, k_2, k_3, \dots$  of  $k$ .

For if  $L$  in (1) lies on  $O$  then

$$NL \Sigma a \cdot MA + LM \Sigma a \cdot NA = 0 \dots \dots \dots (2)$$

$$\text{or} \quad \frac{\Sigma a \cdot MA}{\Sigma a \cdot NA} = \frac{ML}{NL} = \frac{MO}{NO} \quad (\text{Euc. VI. 4.})$$

hence the values of the summation corresponding to any point is proportional to the distance of that point from the central line or axis.

Otherwise thus :—If  $M_1$  and  $N_1$  are the loci of  $M$  and  $N$  such that  $\Sigma a \cdot MA = k_1$  and  $\Sigma a \cdot NA = k_2$  and  $P$  if possible their point of intersection ; then since  $P$  is on both lines  $\Sigma a \cdot PA = k_1$  and  $\Sigma a \cdot PA = k_2$ , which is absurd ; therefore, etc.

**58. Problem.**—*To find the Central Axis  $O$  of a given system of lines  $A, B, C, \dots$  for a given system of multiples  $a, b, c, \dots$*

Take any three points  $P, Q, R$ , and calculate  $\Sigma a \cdot PA$ ,  $\Sigma a \cdot QA$ , and  $\Sigma a \cdot RA$ .

On  $QR$  find a point  $L$  such that

$$\frac{\Sigma a \cdot QA}{\Sigma a \cdot RA} = \frac{QL}{RL}.$$

$L$  is by (2) on the required line ; similarly obtaining points  $M$  and  $N$  on the other sides of the triangle  $P, Q, R$ , their line of connection is that required.

59. Let the multiples  $a, b, c \dots$  denote segments of the given lines  $A, B, C \dots$  respectively;  $a \cdot LA, b \cdot LB, c \cdot LC \dots$  are each twice the area of the triangle subtended by the corresponding segment at the point  $L$ ; hence, *the locus of a point such that the sum of the areas subtended at it by any number of finite lines is constant, ( $k$ ) is a right line*; and if different values be assumed for  $k$  the locus varies in position by moving parallel to itself.

60. **Theorem.**—*The locus of the mean centre  $O$  of the points of intersection  $A_1, B_1, C_1, D_1$ , of a variable line  $L$ , moving parallel to itself, with the sides of a given polygon is a right line.*

Let  $a, b, c$ , etc., be the given multiples and  $\alpha, \beta, \gamma \dots$  the angles at  $A_1, B_1, C_1 \dots$  made by the variable line with the sides  $A, B, C \dots$  of the given polygon.

By hyp.  $\Sigma a \cdot A_1O = 0$ ,

but  $A_1O = OA/\sin \alpha$ ;  $B_1O = OB/\sin \beta$ ;  $C_1O = OC/\sin \gamma$ , etc., substituting these values,

$$a/\sin \alpha \cdot OA + b/\sin \beta \cdot OB + c/\sin \gamma \cdot OC + \text{etc.} = 0,$$

hence  $O$  describes a line, viz., the central axis of the system for the multiples  $a \operatorname{cosec} \alpha, b \operatorname{cosec} \beta, c \operatorname{cosec} \gamma \dots$

**Def.** This locus of the mean centre for the system of parallels, is termed a *Diameter of the Polygon* when the multiples  $a=b=c=\dots=1$ ; a name suggested by the property to which the theorem is reducible when the polygon becomes a circle.

61. **Problem.**—*To find a point  $P$  such that for any systems of lines  $A, B, C \dots$  and multiples  $a, b, c \dots$*

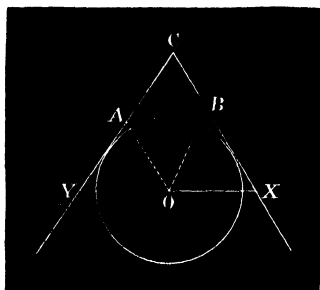
$$\Sigma a \cdot PA^2 \text{ is a minimum.}$$

Let any line  $L$  through  $P$  meet the sides of the polygon in  $A', B', C' \dots$  at angles  $\alpha, \beta, \gamma \dots$ . Then  $\Sigma a \cdot PA^2$

is a minimum when  $\Sigma a \sin^2 \alpha \cdot PA^2$  is a minimum, that is when  $P$  is the mean centre of  $A', B', C' \dots$  for the multiples  $a \sin^2 \alpha, b \sin^2 \beta \dots$ . As  $L$  varies parallel to itself the locus of  $P$  is a diameter. Let it meet the sides of the polygon in  $A_1, B_1, C_1 \dots$ ; the mean centre of these points for the multiples  $a \sin^2 \alpha, b \sin^2 \beta \dots$  is obviously the point required.

## EXAMPLES.

1. If a line is drawn through  $O$  the centre of an escribed circle to meet the sides in  $X$  and  $Y$  such that  $CX=CY$ ; prove that  $AY \cdot BX = (\frac{1}{2}XY)^2$ ; and conversely, if  $AY \cdot BX = (\frac{1}{2}XY)^2$ ,  $AB$  is a tangent to the circle.



[The angles of the triangles  $BOX$  and  $AOY$  are as follows:—

$$BXO = 90 - \frac{1}{2}C, \quad OBX = 90 - \frac{1}{2}B, \quad \text{therefore } BOX = 90 - \frac{1}{2}A;$$

$$AYO = 90 - \frac{1}{2}C, \quad OAY = 90 - \frac{1}{2}A, \quad \text{therefore } AOY = 90 - \frac{1}{2}B.$$

Hence they are similar; therefore, etc.]

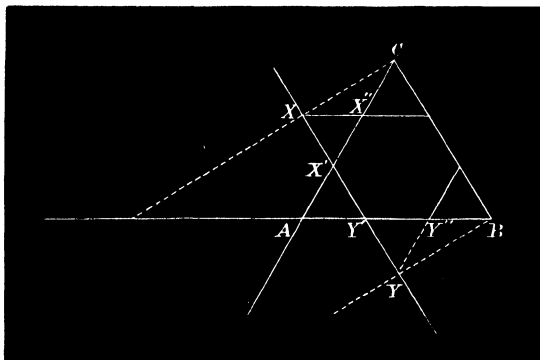
2. The diameters of an equilateral triangle envelope the in-circle.

[Suppose the multiples to be equal to unity, through  $B$  and  $C$  draw any two parallel lines terminated by the opposite sides of the triangle and trisect them in  $X$  and  $Y$  towards the vertices. Since  $X$  and  $Y$  are the mean centres of their intersections with the sides,

the line  $XY$  is a diameter. Draw parallels  $XX''$ ,  $YY''$  to the sides  $AB$  and  $AC$  respectively.

Then the triangles  $XX'X''$  and  $YY'Y''$  are similar, therefore

$$X'X'' \cdot Y'Y'' = XX'' \cdot YY''.$$

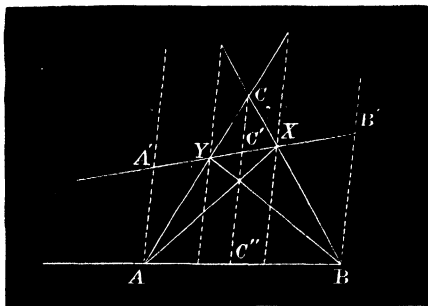


Again, the triangles  $CXX''$  and  $BY Y''$  are similar, since the sides are parallel, therefore

$$XX'' \cdot YY'' = CX'' \cdot BY'' = \left(\frac{1}{2}X''Y''\right)^2,$$

therefore  $X'X'' \cdot Y'Y'' = \left(\frac{1}{2}X''Y''\right)^2$ ;

therefore, etc., by Ex. 1. M'Vicker.]



Otherwise thus :—Draw any system of parallels  $AA'$ ,  $BB'$ ,  $CC'$  terminated by the opposite sides and let  $A'$ ,  $B'$ ,  $C'$  denote the mean

centres of their points of intersection with the sides of  $ABC$ . Let the diameter  $A'B'C'$  meet the sides in  $X, Y, Z$ ; the parallels through  $X$  and  $Y$  are bisected at these points, hence  $AX$  and  $BY$  each bisect  $CC'$  and therefore meet at its middle point. Then from the complete quadrilateral  $ABCXYZ$  the row  $AC''BZ$  is harmonic, therefore  $AA', C'C''$  and  $BB'$  are in harmonic progression, or

$$\frac{1}{AA'} + \frac{1}{BB'} = \frac{2}{C'C''} = \frac{1}{CC''}$$

but  $\Sigma \frac{1}{AA'} = 0$  is the criterion for the tangent to the in-circle. See Art. 55, Ex. 8; therefore, etc.]

3. If a system of  $n$  points  $A, B, C, \dots, N$  be situated at equal distances on an arc of a circle  $O, r$ ; required to find the position of their mean centre.

[Through  $O$  draw a parallel  $L$  to the chord of the arc  $AN$ ; let the angle  $AOL = \alpha$  and  $AON = n\beta$ . Then, if  $d$  be the distance of the mean centre from  $O$ , we have (Art. 53)

$$\begin{aligned} nd &= R \{ \sin \alpha + \sin \alpha + \beta + \sin \alpha + 2\beta + \dots + \sin \alpha + n-1\beta \} \\ &= \frac{\sin(\alpha + \frac{1}{2}n-1\beta) \sin \frac{1}{2}n\beta}{\sin \frac{1}{2}\beta}; \end{aligned}$$

but  $\alpha + \frac{1}{2}n\beta = \frac{1}{2}\pi$ , therefore the above expression becomes, on reduction,  $r \cot \frac{1}{2}\beta \sin \frac{1}{2}n\beta$ .]

NOTE.—If the number of points on the arc is infinitely great, it follows, since  $\beta$  is indefinitely small, that

$$d = \frac{\text{chord} \times \text{radius}}{\text{length of arc}}.$$

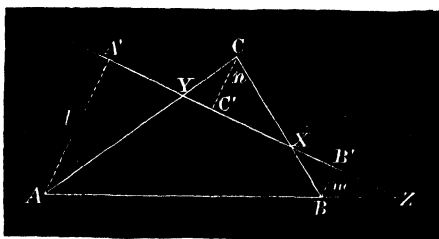
## CHAPTER V.

### COLLINEAR POINTS AND CONCURRENT LINES.

**62. Theorem.**—*If a straight line be drawn cutting the sides of a triangle  $ABC$  in points  $X, Y, Z$ , to prove the relation*

$$\frac{BX}{CX} \cdot \frac{CY}{AY} \cdot \frac{AZ}{BZ} = 1;$$

*and conversely, having given this relation to prove the points are collinear. (Menelaus.)*



For denoting the perpendiculars from the vertices on the transversal by  $l, m, n$ ; we have by similar pairs of triangles,

$$\frac{BX}{CX} = \frac{m}{n}; \quad \frac{CY}{AY} = \frac{n}{l}; \quad \frac{AZ}{BZ} = \frac{l}{m}.$$

Multiplying these equations\* and reducing, the above result follows at once.

Conversely, if the line joining  $X$  and  $Y$  meet the base in  $Z'$  by the first part of the Proposition,

$$\frac{BX}{CX} \cdot \frac{CY}{AY} \cdot \frac{AZ'}{BZ'} = 1,$$

but by hyp.  $\frac{BX}{CX} \cdot \frac{CY}{AY} \cdot \frac{AZ}{BZ} = 1,$

hence  $\frac{AZ}{BZ} = \frac{AZ'}{BZ'},$

therefore  $Z$  and  $Z'$  coincide.

**63. Theorem.**—If three lines  $AO$ ,  $BO$ ,  $CO$  be drawn from the vertices of a triangle  $ABC$  through any point  $O$  to meet the opposite sides in  $X$ ,  $Y$ ,  $Z$ ; to prove the relation

$$\frac{BX}{CX} \cdot \frac{CY}{AY} \cdot \frac{AZ}{BZ} = -1, \dagger$$

and conversely, if this relation be given the lines  $AX$ ,  $BY$ ,  $CZ$  are concurrent. (Ceva.)

For the triangles  $BOC$  and  $COA$  on a common base are proportional to their altitudes, which are in the ratio  $BZ/AZ$ .

\* The proof here given applies equally to the general proposition :— Any right line meeting the sides of a polygon  $ABCDEF \dots$  in points  $X$ ,  $Y$ ,  $Z$ ,  $U$ ,  $V$ ,  $W \dots$  gives the relation

$$\frac{AX}{BX} \cdot \frac{BY}{CY} \cdot \frac{CZ}{DZ} \cdot \frac{DU}{EU} \cdot \frac{EV}{FV} \cdot \frac{FW}{GW} \dots = 1.$$

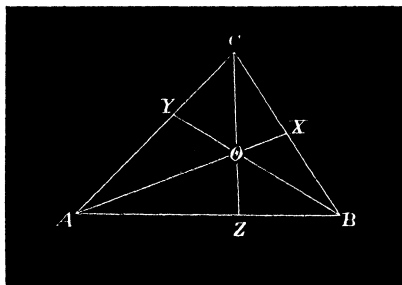
† A line drawn across the sides of a triangle meets them either all externally, or two internally and one externally, i.e. the number of sides cut externally is always *odd*, and therefore the product of the ratios  $\frac{BX}{CX} \cdot \frac{CY}{AY} \cdot \frac{AZ}{BZ}$  is positive. On the other hand, if three points on the sides connect concurrently with the opposite vertices, an *odd* number is *internal* and the product of the ratios is therefore *negative*.



Hence the following equations:—

$$\frac{BX}{CX} = \frac{AOB}{AOC}, \quad \frac{CY}{AY} = \frac{BOC}{BOA}, \quad \frac{AZ}{BZ} = \frac{COA}{COB},$$

on multiplying\* and reducing, the above result is obtained.



Conversely, let  $AX$  and  $BY$  meet in  $O$ . Join  $CO$  and let it meet  $AB$  in  $Z'$ . Then by what has been proved

$$\frac{BX}{CX} \cdot \frac{CY}{AY} \cdot \frac{AZ'}{BZ'} = -1,$$

but by hyp.  $\frac{BX}{CX} \cdot \frac{CY}{AY} \cdot \frac{AZ}{BZ} = -1,$

therefore the points  $Z$  and  $Z'$  coincide.

64. The relations of the previous Articles are equivalent to the two following:—

$$\frac{\sin BAX}{\sin CAX} \cdot \frac{\sin CBY}{\sin ABY} \cdot \frac{\sin ACZ}{\sin BCZ} = \pm 1.$$

For by the rule of sines  $\frac{BX}{CX} = \frac{c \sin BAX}{b \sin CAX},$

with similar values for the remaining ratios, compounding and reducing, the above results are obtained.

---

\* More generally, if the vertices of a polygon  $ABCD\dots$  of any odd number of sides be joined to any point  $O$  and the lines produced to meet the opposite sides in  $X, Y, Z, U, V, W$ , it follows by similar reasoning that

$$\frac{AX}{BX} \cdot \frac{BY}{CY} \cdot \frac{CZ}{DZ} \cdot \frac{DU}{EU} \dots = -1.$$

These formulae may be regarded as criteria of points on the sides of a triangle lying on a line and connecting concurrently with the opposite vertices.

We shall now apply them to the following remarkable particular cases:—

I. Let the points  $X, Y, Z$  be at infinity on the sides, thus  $BX = CX, CY = AY$ , and  $AZ = BZ$ ; hence the criterion of Art. 62 is satisfied and it follows that *every three and therefore all points at infinity in the same plane may be regarded as lying on a line.\**

II. Let  $AX, BY$  and  $CZ$  be any three parallel lines. Since

$$\frac{BX}{CX} \cdot \frac{CY}{AY} \cdot \frac{AZ}{BZ} = -1,$$

*every three, and therefore all, parallel lines are concurrent.*

Of these properties Townsend says: "Paradoxical as these conclusions appear when first stated, all doubt of their legitimacy has been long set at rest by the number and variety of the considerations tending to verify and confirm them."—*Modern Geometry*, Vol. I., Art. 136.

III. When  $AC = BC$ , and  $O$  is a point on the circle touching the equal sides at  $A$  and  $B$ .

By Euc. III. 32,  $\angle BAO = \angle CBO$ ;  $\angle ABO = \angle CAO$ .

Substituting in the above equation, and

$$\frac{\sin ACO}{\sin BCO} = \frac{\sin^2 ABO}{\sin^2 BAO} = \frac{AO^2}{BO^2}.$$

---

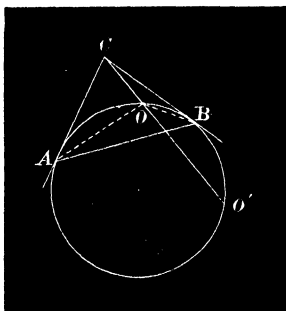
\* This conception of elements situated at an infinite distance is due to Desargues. About the year 1640 he showed that parallel straight lines meet at an infinitely distant point; and that parallel planes may be regarded as intersecting in the line at infinity. More recently the celebrated Poncelet proved that all points at infinity may be considered to lie in a plane.

Similarly, if  $CO$  meet the circle again in  $O'$ ,

$$\frac{\sin ACO}{\sin BCO} = \frac{AO^2}{BO^2}$$

Hence:—*A variable chord  $OO'$  of a circle passing through a fixed point  $C$  divides harmonically the arc  $AB$ , intercepted by the tangents  $CA$  and  $CB$ .*

Also, since  $AB$  is divided harmonically at  $O$  and  $O'$ ,  $OO'$  is divided harmonically by  $AB$ ; hence the variable pairs of tangents at  $O$  and  $O'$  intersect on the fixed line  $AB$ .



IV. Describe a circle about  $AOB$ , and let it meet the lines  $AC$ ,  $BC$ ,  $CO$  again in  $A'B'O'$ .

Then, for the point  $O$ ,

$$\frac{\sin BAO}{\sin ABO} \cdot \frac{\sin CBO}{\sin CAO} = \frac{\sin BCO}{\sin ACO};$$

but  $CBO = CO'B$  and  $CAO = CO'A'$ . (Euc. III. 22.)

Substituting these values and reducing by rule of sines,

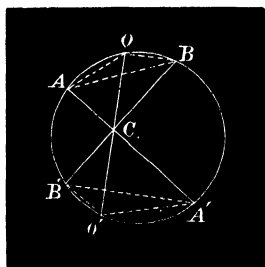
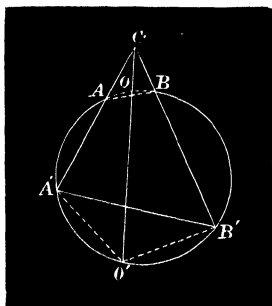
$$\frac{OB}{OA} \cdot \frac{OB'}{OA'} = \frac{\sin BCO}{\sin ACO} \dots \dots \dots (1)$$

Similarly, for  $O'$ ,

$$\frac{O'B}{O'A} \cdot \frac{O'B'}{O'A'} = \frac{\sin BCO}{\sin ACO} \dots \dots \dots (2)$$

Equating these values,

$$\frac{AO}{BO} \div \frac{AO'}{BO'} = \frac{A'O'}{B'O'} \div \frac{A'O}{B'O} * \dots\dots\dots(3)$$



Hence:—If two arcs of a circle  $AB$  and  $A'B'$  are divided in  $O$  and  $O'$  so as to fulfil the relation (3),  $AA'$ ,  $BB'$  and  $OO'$  are concurrent.

#### EXAMPLES.

1. The internal bisectors of the angles of a triangle are concurrent.
2. Any two external and the internal bisector of the remaining angle are concurrent.
3. The lines joining the vertices ( $\alpha^\circ$ ) to the points of contact of the in-circle ( $\beta^\circ$ ) to the internal points of contact of the ex-circles, are concurrent.

[The centres of perspective are named respectively† *point de Gergonne* and *point de Nagel* of the triangle.]

---

\* The function  $\frac{AO}{BO} \div \frac{AO'}{BO'}$  is termed the Anharmonic Ratio of the points  $A, B, O, O'$ ; and (3) may be expressed thus:—"If the arcs  $AB$  and  $A'B'$  are divided *equi-anharmonically* in  $O$  and  $O'$ , the lines  $AA'$ ,  $BB'$  and  $OO'$  are concurrent; and conversely."

† *Educational Times*, July, 1890.

4. The perpendiculars of a triangle are concurrent.

5. The tangents to the circum-circle at  $A, B, C$  meet the opposite sides collinearly.

6. If a circle meet the sides of a triangle in  $X, X', Y, Y', Z, Z'$  such that either triad  $X, Y, Z$  is collinear or connects concurrently with the opposite vertices; a similar relation exists amongst the remaining points  $X', Y', Z'$ .

7. If three points are collinear, their *isotomic conjugates* with respect to the sides are collinear.

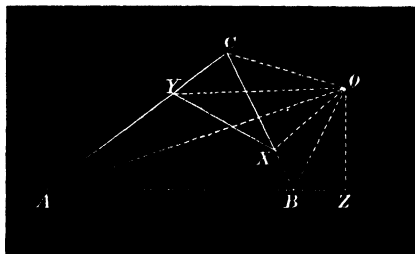
7a. If they connect concurrently with the vertices, their *isogonal conjugates* with respect to the angles also connect concurrently.

8. For any triangle  $ABC$  and transversal  $XYZ$ ; if any point  $O$  is joined to the six points

$$\frac{\sin BOX}{\sin COX} \cdot \frac{\sin COY}{\sin AOY} \cdot \frac{\sin AOZ}{\sin BOZ} = 1.*$$

[For  $\frac{BX}{CX} = \frac{BO \sin BOX}{CO \sin COX}$ , with similar values for  $\frac{CY}{AY}$  and  $\frac{AZ}{BZ}$ ; therefore, etc. ...]

9. If the sides of a triangle and any three concurrent lines



\* Examples 8 and 9 will be afterwards enunciated as follows :

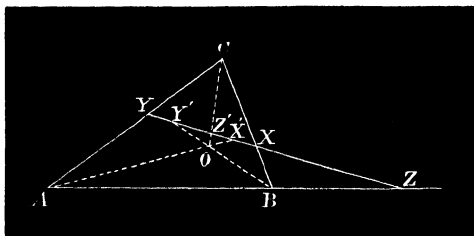
8°. The lines joining any point to the six vertices of a quadrilateral form a pencil of rays in Involution.

9°. Any line drawn across the sides and diagonals of a quadrilateral is cut in Involution.

through its vertices are cut by a transversal in six points  $X$  and  $X'$ ,  $Y$  and  $Y'$ ,  $Z$  and  $Z'$ ; ( $BC$  in  $X$ ,  $AO$  in  $X'$  ...)

$$\frac{YX'}{ZX'} \cdot \frac{ZY'}{XY'} \cdot \frac{XZ'}{YZ'} = 1,*$$

and conversely.



[For  $\frac{\sin BAO}{\sin CAO} = \frac{ZX'}{YX'}$  with similar values for  $\frac{\sin CBO}{\sin ABO}$  and  $\frac{\sin ACO}{\sin BCO}$ ; therefore, etc.]

10. If  $AX$ ,  $BY$ ,  $CZ$  are concurrent, the intersections of  $YZ$  and  $BC$  ( $X'$ ),  $ZX$  and  $CA$  ( $Y'$ ),  $XY$  and  $AB$  ( $Z'$ ) are collinear.

[For  $\frac{BX'}{CX'} \cdot \frac{CY'}{AY'} \cdot \frac{AZ'}{BZ'} = 1$ . Compounding this with two similar equations involving  $Y'$  and  $Z'$  and reducing, we have

$$\frac{BX'}{CX'} \cdot \frac{CY'}{AY'} \cdot \frac{AZ'}{BZ'} = 1.]$$

11. Given two points  $A$  and  $B$  on a circle  $MNP$ , on the same side of the diameter  $MN$ ; find a point  $P$  on the other side such that the intersections  $X$  and  $Y$  of  $AP$  and  $BP$  respectively with  $MN$  may be equidistant from the centre.

[Let  $AB$  and  $MN$  meet in  $Z$ ; then it is easily proved that  $PX^2/PY^2 = BZ/AZ$ ; hence the species of the triangle  $PXY$  is known; therefore, etc.]

12. Draw two circles in contact each touching a given line at a given point and having their radii in a given ratio.

---

\* Will be afterwards seen to be an *Equation of Involution* of the pencil.

13. If lines be drawn from the vertices of  $ABC$  to a point  $\Omega$  such that  $\Omega BC = \Omega CA = \Omega AB = \theta$ , prove that  $\theta$  is given by the equation  

$$\cot \theta = \cot A + \cot B + \cot C.$$

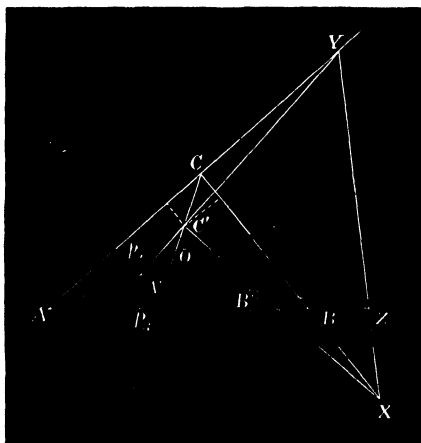
[For  $\sin^3 \theta = \sin(A - \theta) \sin(B - \theta) \sin(C - \theta)$ ; etc. Cf. Art. 28.]

14. In the general case if the lines in Ex. 13 making equal angles ( $\alpha$ ) with the sides are not concurrent, they form a triangle  $A'B'C'$  similar to  $ABC$  and the ratio of similitude is equal to  

$$\cos \alpha - \sin \alpha (\cot A + \cot B + \cot C) : 1.$$

**Defs.** The *Centres of Perspective* of two lines  $AB$  and  $A'B'$  are the points of intersection of the pairs of lines  $AB', A'B$  and  $AA', BB'$  joining their extremities.

Two triangles are said to be in perspective when the lines joining corresponding vertices meet in a point. This point is called the *Centre of Perspective* of the triangles.



### 65. Criterion of Perspective of Triangles. Theorem.

If the perpendiculars from the vertices of a triangle  $A'B'C'$  on the sides of another  $ABC$  be denoted by  $p_1, p_2, p_3$ ;

$q_1, q_2, q_3; r_1, r_2, r_3$  (i.e. from  $A'$  on  $BCp_1$ ,  $A'$  on  $CAp_2$ , and so on), the two are in perspective if

$$\frac{q_1}{r_1} \cdot \frac{r_2}{p_2} \cdot \frac{p_3}{q_3} = 1; \text{ and conversely.}$$

For let  $AA'$  meet  $BC$  in  $X'$ . Then

$$\sin BAX' / \sin CAX' = p_3 / p_2,$$

with similar values for  $r_2/r_1$ , and  $q_1/q_3$ ; multiplying these equations together, therefore, etc., by Art. 64, which also proves the converse\* proposition.

**66. Theorem.**—*If the vertices of two triangles connect concurrently, their pairs of corresponding sides intersect collinearly ( $BC$  and  $B'C'$  in  $X$ , etc. ...).*

For, by similar triangles,

$$\frac{q_1}{r_1} = \frac{B'X}{C'X}, \quad \frac{r_2}{p_2} = \frac{C'Y}{A'Y}, \quad \text{and} \quad \frac{p_3}{q_3} = \frac{A'Z}{B'Z}$$

Multiplying, we have

$$\frac{B'X}{C'X} \cdot \frac{C'Y}{A'Y} \cdot \frac{A'Z}{B'Z} = \frac{q_1}{r_1} \cdot \frac{r_2}{p_2} \cdot \frac{p_3}{q_3} = 1, \text{ therefore, etc.}$$

**Def.** The line of collinearity is termed the *Axis of Perspective* or *Homology*† of the triangles.

#### EXAMPLES.

1. Any triangle escribed to a circle is in perspective with that formed by joining the points of contact of its sides.

[The centre of perspective is the symmedian point of the inscribed triangle.]

\* Or thus :—Let  $O$  be the centre of perspective of the triangles and  $\alpha, \beta, \gamma$  the perpendiculars from it on the sides of  $ABC$ ; since  $\beta/\gamma = p_2/p_3$ ,  $\gamma/\alpha = q_3/q_1$ , and  $\alpha/\beta = r_1/r_2$ ; multiply and reduce; therefore, etc.

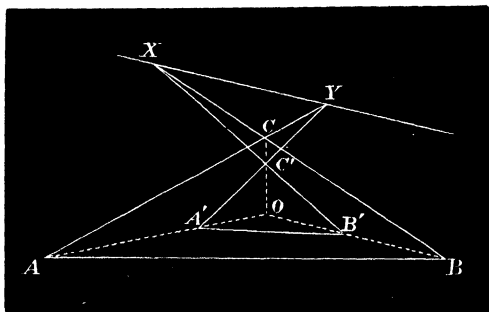
† The term Homology is due to Poncelet who first studied the properties of homological figures in space, v. *Traité des propriétés projectives des figures* (1822).



2. If three triangles  $ABC$ ,  $A_1B_1C_1$ ,  $A_2B_2C_2$  have a common axis of perspective  $XYZ$ , their centres of perspective when taken two and two are collinear.

[For the triangles (fig. of Ex. 3)  $BB_1B_2$  and  $CC_1C_2$  are in perspective, their centre being at  $X$ ; similarly  $Y$  is the centre of perspective of  $CC_1C_2$ ,  $AA_1A_2$  and  $Z$  of  $AA_1A_2$  and  $BB_1B_2$ . Hence the corresponding sides of these pairs of triangles intersect in collinear points. But these points (e.g.  $AA_1$ ,  $BB_1$ ) are the centres of perspective of the given triangles in pairs; therefore, etc.]

3. If three triangles  $ABC$ ,  $A_1B_1C_1$ ,  $A_2B_2C_2$  have a common centre of perspective, their axes are concurrent.



[Consider the three triangles whose sides are respectively the directions  $BC$ ,  $B_1C_1$ ,  $B_2C_2$ ;  $CA$ ,  $C_1A_1$ ,  $C_2A_2$ ;  $AB$ ,  $A_1B_1$ ,  $A_2B_2$ .

It is manifest they are in pairs in perspective, the axis of the first pair being  $CC_1$ ; and  $XY$  is a line joining corresponding vertices.

Thus the axis of perspective  $XY$  of any two and therefore of every two of the given triangles passes through the centre of perspective of the conjugate triad]

NOTE.—It will be noticed that the common centre  $O$  of the three given triangles is the point of concurrence of the axes  $AA_1$ ,  $BB_1$ ,  $CC_1$  of the conjugate triad, and the common centre of the conjugate triad taken in pairs is the point of concurrence of the axes of the given triangles.

4. Brocard's first triangle is in perspective in three ways with  $ABC$ .

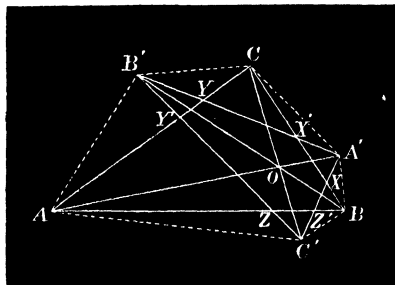
[The Brocard points are evidently two centres of perspective (Art. 28); also the lines  $AA'$ ,  $BB'$ ,  $CC'$  are concurrent, for  $p_2/p_3$  found by aid of the property of Art. 28, Ex. 2, to be  $c^3/b^3$ ; therefore, etc.

The three centres of perspective are the mean centres of the vertices  $ABC$  for multiples proportional to (Art. 52)

$$\frac{1}{a^2}, \frac{1}{b^2}, \frac{1}{c^2}; \frac{1}{b^2}, \frac{1}{c^2}, \frac{1}{a^2}; \frac{1}{c^2}, \frac{1}{a^2}, \frac{1}{b^2}]$$

5. If  $\Omega$ ,  $\Omega'$ ,  $\Omega''$  denote the three centres of perspective of  $ABC$  and its first Brocard triangle  $A'B'C'$ , to prove that the corresponding vertices of their three median triangles lie on three right lines. (Stoll.)

[For  $A'B'C'$  and  $ABC$  have a common centroid  $G$  (Art. 53, Ex. 6). But  $\Omega\Omega'\Omega''$  has the same centroid; for its vertices are the mean centres of  $A$ ,  $B$ ,  $C$  for multiples proportional to  $\frac{1}{b^2}, \frac{1}{c^2}, \frac{1}{a^2}$ ;  $\frac{1}{c^2}, \frac{1}{a^2}, \frac{1}{b^2}$ ;  $\frac{1}{a^2}, \frac{1}{b^2}, \frac{1}{c^2}$ ; therefore (Art. 53, Cor. 3) the mean centre of  $\Omega$ ,  $\Omega'$ ,  $\Omega''$  is that for  $A$ ,  $B$ ,  $C$  for multiples each  $=1/a^2+1/b^2+1/c^2$ . Now let  $L$ ,  $L'$ ,  $L''$  be the middle points of the corresponding sides of the three triangles such that  $GA=2GL$ ,  $GA'=2GL'$ , and  $G\Omega''=2GL''$ : since  $A$ ,  $A'$ ,  $\Omega''$  are collinear;  $L$ ,  $L'$ ,  $L''$  are also collinear, and the two lines of collinearity parallel.]



**67. Theorem.**—Two triangles  $ABC$  and  $A'B'C'$  are in perspective when

$$\frac{BX \cdot BX'}{CX \cdot CX'} \cdot \frac{CY \cdot CY'}{AY \cdot AY'} \cdot \frac{AZ \cdot AZ'}{BZ \cdot BZ'} = 1,$$

where  $X$  and  $X'$  are the points of intersection of  $BC$  with  $C'A'$  and  $A'B'$ , etc.; and conversely.

Using the previous notation, we have by similar triangles

$$\frac{BX}{CX} = \frac{q_2}{r_2}, \frac{BX'}{CX'} = \frac{q_3}{r_3}, \text{ etc.}$$

Hence

$$\frac{BX \cdot BX'}{CX \cdot CX'} = \frac{q_2 q_3}{r_2 r_3};$$

therefore the left side of the above equation becomes

$$\frac{q_2 q_3}{r_2 r_3} \cdot \frac{r_3 r_1}{p_3 p_1} \cdot \frac{p_1 p_2}{q_1 q_2},$$

which is equal to, on reduction,

$$\frac{r_1}{q_1} \cdot \frac{p_2}{r_2} \cdot \frac{q_3}{p_3}; \text{ therefore, etc. (Art. 65.)}$$

**COR. 1. Pascal's Theorem.**—If  $XX'YY'ZZ'$  be any cyclic hexagon, then (Euc. III. 36)

$$AY \cdot AY' = AZ \cdot AZ'; \quad BZ \cdot BZ' = BX \cdot BX', \text{ etc.}$$

Hence:—*The two triangles formed by the two triads of alternate sides of any cyclic hexagon are in perspective; or, the opposite sides of a cyclic hexagon meet in three collinear points.*

The centre and axis of perspective of any two triangles in perspective are called the *Pascal\* Point* and *Line* of the hexagon  $XX'YY'ZZ'$ , which is termed a *Pascal Hexagon*.

**COR. 2.** If  $X, X'$ ;  $Y, Y'$ ;  $Z, Z'$  coincide in pairs on the circle, the sides of the hexagon become the tangents to the circle at  $X, Y, Z$ , and the chords of contact  $YZ, ZX$  and  $XY$ ; the Pascal point is therefore the symmedian point of the triangle  $XYZ$ . (Art. 66, Ex. 1.)

---

\* When only sixteen years old, Pascal discovered this property of the mystic hexagram. *Essai sur les Coniques*, Pascal, 1640.

COR. 3.

$$\frac{\sin BA'X \sin CA'X}{\sin BA'X' \sin CA'X'} \cdot \frac{\sin CB'Y \sin AB'Y}{\sin CB'Y' \sin AB'Y'} \cdot \frac{\sin AC'Z \sin BC'Z}{\sin AC'Z' \sin BC'Z'} = 1.$$

[For  $\frac{\sin BA'X}{\sin BA'X'} = \frac{q_2}{q_3}$ ;  $\frac{\sin CA'X}{\sin CA'X'} = \frac{r_2}{r_3}$ , etc. ;

hence the above expression is equivalent to

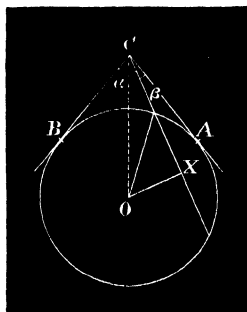
$$\frac{q_2 r_2}{q_3 r_3} \cdot \frac{r_3 p_3}{r_1 p_1} \cdot \frac{p_1 q_1}{p_2 q_2} = \frac{q_1}{r_1} \cdot \frac{r_2}{p_2} \cdot \frac{p_3}{q_3} = 1.]$$

COR. 4. **Brianchon's Theorem.**—Let  $AC'BA'CB'$  be an escribed hexagon and  $x, y, z$  the intercepts made by the circle on the sides of the triangle  $A'B'C'$ ; since

$$\frac{\sin BA'X \sin CA'X}{\sin BA'X' \sin CA'X'} = \frac{y^2}{z^2} *$$

with two other similar equations, Cor. 3 in this particular

\*The property on which this depends is as follows:—If from the point of intersection  $C$  of two tangents  $CA, CB$  to a circle a secant of length  $x$  is drawn dividing the angle  $ACB$  into segments  $\alpha$  and  $\beta$ ; then  $\sin \alpha \sin \beta \propto x^2$ .



For if  $O$  be the centre of the circle and  $OX$  a perpendicular to the secant, we have

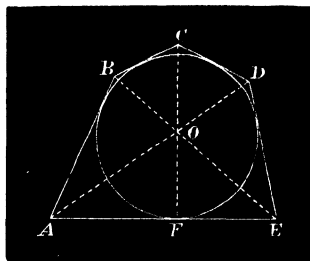
$$\sin \alpha \sin \beta = \sin^2 \frac{1}{2}(\alpha + \beta) - \sin^2 \frac{1}{2}(\alpha - \beta) = r^2/OC^2 - OX^2/OC^2 = x^2/4OC^2;$$

therefore, etc.

case reduces to:—*The lines connecting the opposite vertices of an escribed hexagon are concurrent; or, the two triangles formed by joining the alternate vertices of an escribed hexagon are in perspective.*

The centre and axis of perspective of the triangles are termed the *Brianchon \* Point* and *Line* of the hexagon  $AC'BA'CB'$ , which for the same reason is called a *Brianchon Hexagon*.

COR. 5. If two of the sides  $AF$  and  $EF$  of an escribed hexagon coincide, the vertex  $F$  is the point of contact of the tangent  $AE$  (Art. 6); *hence for an escribed pentagon  $ABCDE$ , if the lines  $AD$  and  $BE$  meet in  $O$ , the points  $C, O, F$  are collinear* (cf. Art. 63, foot-note).



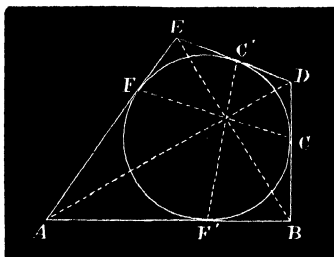
COR. 6. If two pairs of sides  $BC, CD$  and  $AF, EF$  coincide, the hexagon reduces to a quadrilateral  $ABDE$ ; hence the diagonals  $AD$  and  $BE$  meet on  $CF$ ; similarly they meet on  $C'F'$ ; therefore *the internal diagonals of an escribed quadrilateral and of the corresponding inscribed meet in a point.*

---

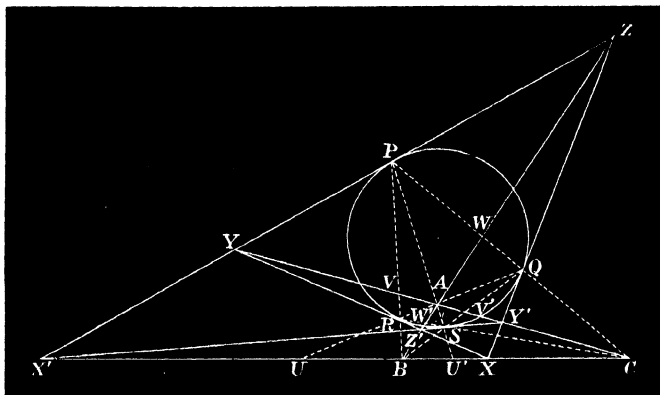
\* Published by Brianchon in the year 1806, and derived by him from Pascal's Theorem by the process of reciprocation with respect to the circle. (See Art. 80, 2°.)

COR. 7. Consider the cyclic hexagon  $FFC'CCF'$ .

Its Pascal line is the line of collinearity of the three points (1)  $FF, CC$ ; (2)  $FC', CF'$ ; (3)  $FF', CC'$ : but the line



joining (2) and (3) is the third diagonal of the inscribed quadrilateral  $CFC'F'$  and (1) is the intersection of the tangents at  $C$  and  $F$ , and therefore one extremity of the third diagonal of the escribed quadrilateral; hence:—*the third diagonals of any inscribed and corresponding escribed quadrilaterals coincide.*



COR. 8. Let  $PQRS$  be any cyclic quadrilateral; and let  $XX'YY'ZZ'$ , the corresponding escribed quadrilateral,

be regarded as a Brianchon hexagon  $ZPX'Z'RX$  whose two pairs of coincident sides are the tangents from  $Y$ . Then the lines  $ZZ'$ ,  $PR$ ,  $XX'$  are concurrent at the Brianchon point  $B$ ; similarly, if the pairs of coincident sides are the tangents from  $Y'$ , we have  $ZZ'$ ,  $QS$ ,  $XX'$  concurrent, i.e. the pairs of opposite connectors  $PR$  and  $QS$  of the inscribed quadrilateral and  $ZZ'$  and  $XX'$  of the corresponding escribed cointersect. We see therefore from Cors. 7 and 8 that *any pair of opposite connectors of an inscribed quadrilateral and the corresponding pair for the quadrilateral escribed at its vertices are concurrent*. The three points of concurrence on the figure are  $A$ ,  $B$ ,  $C$ .

The points  $U$ ,  $V$ ,  $W$ ,  $U'$ ,  $V'$ ,  $W'$  lie in triads on four lines.

#### EXAMPLES.

1. Three pairs of tangents are drawn from the vertices of a triangle to any circle to meet the opposite sides in points  $XX'$ ,  $YY'$ ,  $ZZ'$ ; show that if  $X$ ,  $Y$ ,  $Z$  are collinear,  $X'$ ,  $Y'$ ,  $Z'$  are also collinear.

[Apply Cor. 4.]

2.  $ABC$  is a triangle inscribed in and in perspective with  $A'B'C'$ ; the tangents from  $ABC$  to the in-circle of  $A'B'C'$  meet the opposite sides in three collinear points  $X$ ,  $Y$ ,  $Z$  ( $BC$  in  $X$ , etc.).

[Let the axis of perspective of the two triangles be  $X'Y'Z'$ , therefore by Cor. 4 we have  $\left(\frac{BX \cdot BX'}{CX \cdot CX'}\right)(\dots)(\dots)=1$ ; therefore, etc., by Ex. 1.]

3. If points  $XX'$ ,  $YY'$ ,  $ZZ'$  be taken on the sides of a triangle such that

$$\frac{BX}{CX} \cdot \frac{BX'}{CX'} \cdot \frac{CY}{AY} \cdot \frac{CY'}{AY'} \cdot \frac{AZ}{BZ} \cdot \frac{AZ'}{BZ'} = 1,$$

they are the vertices of a Pascal hexagon.

4. The lines joining each pair of points to the opposite vertex ( $AX$  and  $AX'$ , etc.) of the triangle determine a Brianchon hexagon.

5. ( $\alpha^\circ$ ) Any two transversals  $XYZ$ ,  $X'Y'Z'$  determine on the sides the vertices of a Pascal hexagon.

( $\beta^\circ$ ) Two triads of points on the sides which connect concurrently with the opposite vertices determine a Pascal hexagon.

( $\gamma^\circ$ ) A transversal  $XYZ$  and three points  $X'$ ,  $Y'$ ,  $Z'$  which connect concurrently with the opposite vertices determine a Brianchon hexagon.

6. A hexagon is inscribed in a circle; prove that the continued products of the perpendiculars from any point on the Pascal line on the alternate sides are equal ( $xyz = x'y'z'$ ).

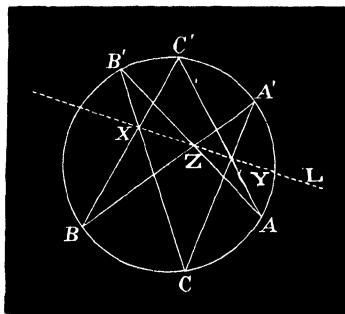
[Let  $AB'CA'BC'$  be the hexagon whose pairs of opposite sides  $BC'$ ,  $B'C$ ;  $CA'$ ,  $C'A$ ;  $AB'$ ,  $A'B$  meet in points  $X$ ,  $Y$ ,  $Z$  respectively and the Pascal line  $L$  ( $XYZ$ ) at angles  $\alpha$ ,  $\alpha'$ ,  $\beta$ ,  $\beta'$ ,  $\gamma$ ,  $\gamma'$ ; then

$$\frac{BL \cdot CL}{B'L \cdot CL} = \frac{BX \cdot CX \sin^2 \alpha}{B'X \cdot CX \sin^2 \alpha'} = \frac{\sin^2 \alpha}{\sin^2 \alpha'} \quad (\text{Euc. III. 36})$$

$$\text{Similarly, } \frac{C'L \cdot AL}{CL \cdot A'L} = \frac{\sin^2 \beta}{\sin^2 \beta'} \text{ and } \frac{AL \cdot BL}{A'L \cdot BL} = \frac{\sin^2 \gamma}{\sin^2 \gamma'}$$

Multiplying these equations and reducing,

$$\sin^2 \alpha \sin^2 \beta \sin^2 \gamma = \sin^2 \alpha' \sin^2 \beta' \sin^2 \gamma'; \text{ therefore, etc.}]$$

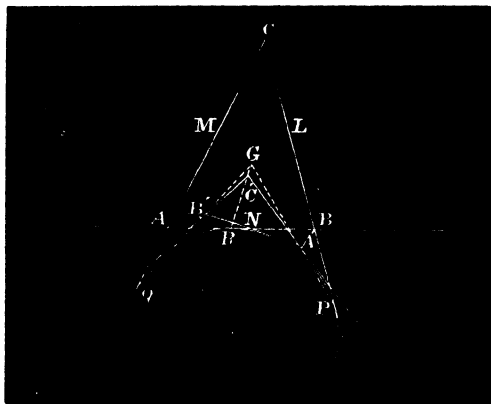


7. From the middle points  $L$ ,  $M$ ,  $N$  of the sides of a triangle tangents are drawn to the in-circle; show that these tangents form a triangle ( $A'B'C'$ ) in perspective with that ( $PQR$ ) obtained by joining the points of contact of the in- or ex-circles with the sides, and the centre of perspective is the median point of  $ABC$ .

[For since the sides of  $ABC$  with any two of the tangents form



an escribed pentagon, *e.g.*,  $BCMNA'$ , by Cor. 5, the lines  $BM$ ,  $CN$ ,  $A'P$  are concurrent; that is,  $A'P$  passes through the centroid ( $BM$ ,  $CN$ ). Similarly for  $B'Q$ ,  $C'R$ ; therefore, etc.]



NOTE.—If  $LMN$  is *any* inscribed triangle in perspective with  $ABC$ , the above reasoning applies to prove that  $A'BC'$  and  $PQR$  have the *same* centre of perspective.

8. If two triangles  $ABC$  and  $A'B'C'$  are in perspective,  $A'BC$ ,  $AB'C$ ;  $AB'C$ ,  $A'BC'$ ;  $ABC'$ ,  $A'B'C$  are also in perspective.

9. If  $AA'$ ,  $BB'$ ,  $CC'$  denote the lengths of three lines whose directions are concurrent, their six centres of perspective (of  $BB'$  and  $CC'$ ,  $X$  and  $X'$ , etc.) taken in pairs lie in triads on four lines.

[For they are the axes of perspective of the triangles in Ex. 8.]

10. If  $X$ ,  $Y$ ,  $Z$  are on the sides of a triangle and fulfil the relation  $\Sigma(BX^2 - CX^2) = 0$ , the perpendiculars through them to the sides are concurrent; and conversely.

11. If two triangles are such that the perpendiculars from the vertices of either upon the sides of the other are concurrent, then conversely the perpendiculars from the vertices of the latter upon the sides of the former are concurrent.

[By Ex. 10.]

12. State the particular cases of the Theorem of Ex. 11 for a given triangle taken with the ( $\alpha^\circ$ ) pedal, ( $\beta^\circ$ ) median, ( $\gamma^\circ$ ) triangles formed by joining the points of contact with the sides of the in- or ex-circles.

13. If  $XYZ$  be a transversal to a triangle  $ABC$ ,  $X'$ ,  $Y'$ ,  $Z'$  the harmonic conjugates of  $X$ ,  $Y$ ,  $Z$ , with respect to the sides; prove that

1°. The triads of points  $Y'Z'X$ ,  $Z'X'Y$ ,  $X'Y'Z$  are collinear.

2°.  $X'Y'Z'$ ,  $X'YZ$ ,  $Y'ZX$ ,  $Z'XY$  connect concurrently with the opposite vertices.

14. The middle points of the segments  $XX'$ ,  $YY'$ ,  $ZZ'$  are collinear.

[For they are the middle points of the diagonals of a complete quadrilateral by Ex. 3. For another proof v. Art. 91.]

15. The perpendiculars from the vertices of a triangle  $ABC$  on the sides of  $A'B'C'$ , its first Brocard triangle, are concurrent on the circum-circle. (Tarry's Point.)

[By the theorem of Ex. 11.]

16. The perpendiculars from the middle points of the sides of  $A'B'C'$  on the sides of  $ABC$  are concurrent. (Cf. Ex. 15.)

17. The Simson line of Tarry's point is perpendicular to  $OK$ , the line joining the circum-centre to the symmedian point.

18. In the figure of Art. 28 show that

$$OA' : OB' : OC' = \cos(A + \omega) : \cos(B + \omega) : \cos(C + \omega);$$

and deduce the formula for the Brocard angle,

$$\sin A \cos(A + \omega) + \sin B \cos(B + \omega) + \sin C \cos(C + \omega) = 0.$$

*Note on Tarry's Point.*—It will appear obvious that the diameter of the circum-circle containing Tarry's point is related to the triangle  $ABC$  in the same manner as  $OK$  is to  $A'B'C'$ ; and that the circum- and Brocard circles are divided similarly by these *corresponding* diameters. Also, if  $\alpha$ ,  $\beta$ ,  $\gamma$  denote the perpendiculars from Tarry's point on the sides of  $ABC$ ,

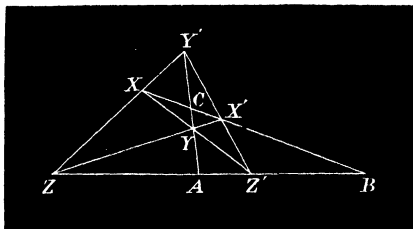
$$\alpha : \beta : \gamma = \sec(A + \omega) : \sec(B + \omega) : \sec(C + \omega).$$

A point of interest may be here noticed. From Art. 28, Ex. 18 (note) it is evident that the centres  $O_1$ ,  $O_2$ ,  $O_3$  of Neuberg's circles with respect to the sides of  $ABC$  are the vertices of similar isosceles

triangles described on  $a, b, c$  respectively, whose equal base angles are  $\frac{1}{2}\pi - \omega$ . Therefore, if  $T$  denote Tarry's point, it easily follows that  $AT, AO_1; BT, BO_2; CT, CO_3$  divide the angles of  $ABC$  isogonally. *But the isogonal conjugate of a point on the circum-circle is at infinity; hence the lines  $AO_1, BO_2, CO_3$  are parallel.*

### HARMONIC PROPERTIES OF THE QUADRILATERAL.

**68. Theorem.**—*In any complete quadrilateral each of the diagonals  $XX', YY', ZZ'$  is divided harmonically by the other two.*



Consider the triangle  $ZZ'Y'$  and transversal  $BXX'$ ,

$$\frac{Z'X'}{Y'X'} \cdot \frac{Y'X}{ZX} = \frac{Z'B}{ZB} \dots \dots \dots (1)$$

And since  $YY', YZ, YZ'$  are three concurrent lines through its vertices, we have

$$\frac{Z'X'}{Y'X'} \cdot \frac{Y'X}{ZX} = -\frac{Z'A}{ZA} \dots \dots \dots (2)$$

Equating these results, we have  $ZA/Z'A = -ZB/Z'B$ .

Hence the row of points  $ZZ'AB$  is harmonic.

Similarly,  $BCXX'$  and  $CAYY'$  are harmonic.

**COR. 1.** The angles of the triangle  $ABC$ , formed by the diagonals (*the diagonal triangle*) are divided harmonically by the pairs of lines  $AX, AX'; BY, BY'; CZ, CZ'$ .

**COR. 2.** If two lines be given in magnitude and position ( $ZZ'$  and  $XX'$ ) their two centres of perspective ( $Y$

and  $Y'$ ) joined to their point of intersection ( $B$ ) form a harmonic pencil. They also divide the line joining their centres of perspective (in  $A$  and  $C$ ) harmonically.

**Problem.**—To determine the number of polygons which can be formed from  $n$  points.

Each point joined to the remaining  $n-1$  points gives  $n-1$  lines. Taking any one of these lines as the first side of the polygon we have similarly  $n-2$  selections for the second side,  $n-3$  for the third side, and so on. Therefore we have  $(n-1)(n-2)$  selections for the first two sides,  $(n-1)(n-2)(n-3)$  for the first three sides, etc.; hence we have finally  $\frac{n-1}{2}$  equal to twice the number of polygons, since any sequence of sides when reversed gives the same polygon.

Thus four points may be joined in three ways as in figure.

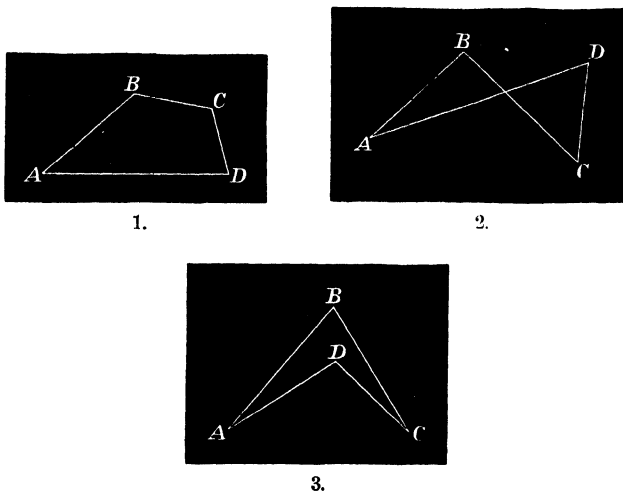


Fig. 1 is called a *Convex*, Fig. 2 an *Intersecting*, and Fig. 3 a *Re-entrant Polygon*.

By application of the general formula to the hexagon we find that six points in general determine a system of sixty hexagons.

## EXAMPLES.

1. The conditions that the quadrilaterals in the figures are escribed are :—

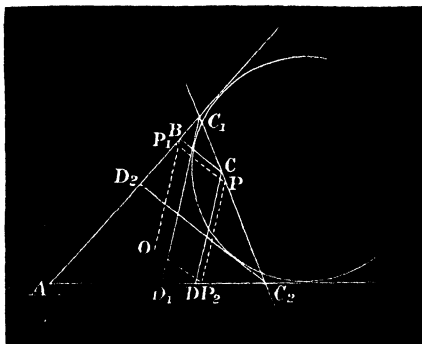
$$1^{\circ}. BC + AD = AB + CD.$$

$$2^{\circ}. BC \sim AD = AB \sim CD.$$

$$3^{\circ}. BC \sim AD = AB \sim CD.$$

[Since tangents from any point to a circle are equal.]

2. To prove that the quadrilateral whose angles and perimeter are given is of maximum area when it is escribed to a circle. (Hermite.)



[Let two of the sides  $AB$  and  $AD$  be fixed in position and the remaining two vary. It is easy to see that the locus of  $C$  is a line. Suppose  $C_1$  and  $C_2$  to be the positions of  $C$  on the fixed lines and  $C_1D_1$ ,  $C_2D_2$  parallels to the fixed directions  $CD$  and  $CB$ .

The perimeters of the triangles  $AC_1D_1$  and  $AC_2D_2$  are each equal to the perimeter of the quadrilateral  $ABCD$ ; the ex-circle of  $AC_1D_1$  is the ex-circle of  $AD_2C_2$  and  $D_2C_2 - D_2C_1 = D_1C_1 - D_1C_2$ .

Now, for any point  $P$  and parallels  $PP_1$  and  $PP_2$  by similar triangles  
 $PC_1/C_1C_2 = (PP_1 - P_1C_1)/(D_2C_2 - D_2C_1)$   
 and  $PC_2/C_1C_2 = (PP_2 - P_2C_2)/(D_1C_1 - D_1C_2)$ ;  
 adding these equations, we get

$$PP_1 + PP_2 - P_1C_1 - P_2C_2 = D_2C_2 - D_2C_1;$$

to each side add  $AC_1 + AC_2$ , and

$$AP_1 + P_1P + PP_2 + P_2A = AD_2 + D_2C_2 + C_2A = \text{given perimeter}.$$

Regarding  $P$  and  $C$  as consecutive points on the locus, the area of the quadrilateral is a maximum when  $BCPP_1 = CDP_2$ , i.e.  $BD$  is parallel to  $C_1C_2$ . Hence the parallels  $BO$  and  $DO$  to  $CD$  and  $BC$  respectively form with  $AB$  and  $AD$  a re-entrant escribed quadrilateral, and therefore  $AB + BO = AD + DO$  or (Euc. I. 34)  $AB + CD = AD + BC$ ; therefore, etc.]

It may at once be inferred that *the maximum polygon of any order, of given angles and perimeter, is escribed to a circle.*

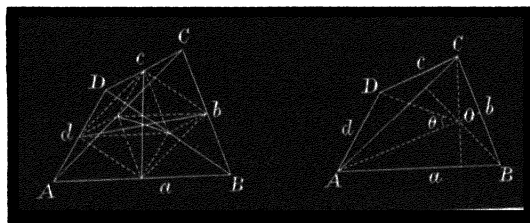
3. If three common tangents  $D, E, F$  to three circles  $A, B, C$  taken two and two are concurrent; prove that the conjugate triad  $D', E', F'$  are also concurrent.\*

4. Let the lines joining the middle points of the three pairs of opposite connectors,  $BC$  and  $AD$ , etc., of four points  $A, B, C, D$  be  $\lambda, \mu, \nu$ ; prove by means of the following evident formulae,

$$4\lambda^2 = \delta^2 + \delta'^2 + 2\delta\delta'\cos\hat{\delta\delta'} = a^2 + c^2 + 2ac\cos\hat{ac}, \dots\dots\dots(1)$$

$$4\mu^2 = b^2 + d^2 - 2bd\cos\hat{bd} = a^2 + c^2 - 2ac\cos\hat{ac}, \dots\dots\dots(2)$$

$$4\nu^2 = \delta^2 + \delta'^2 - 2\delta\delta'\cos\hat{\delta\delta'} = b^2 + d^2 + 2bd\cos\hat{bd}, \dots\dots\dots(3)$$



the relations,

$$1^\circ. 2(\mu^2 + \nu^2) = c^2 + a^2; 2(\nu^2 + \lambda^2) = b^2 + d^2; 2(\lambda^2 + \mu^2) = \delta^2 + \delta'^2:$$

$$2^\circ. 4(\lambda^2 + \mu^2 + \nu^2) = a^2 + b^2 + c^2 + d^2 + \delta^2 + \delta'^2:$$

$$3^\circ. 4\lambda^2 = b^2 + d^2 - c^2 - a^2 + \delta^2 + \delta'^2:$$

with similar expressions for  $\mu$  and  $\nu$ :

$$4^\circ. \mu^2 - \nu^2 = ac\cos\hat{ac}; \nu^2 - \lambda^2 = -bd\cos\hat{bd}; \lambda^2 - \mu^2 = -\delta\delta'\cos\hat{\delta\delta'};$$

$$2(\mu^2 - \nu^2) = \delta^2 + \delta'^2 - b^2 - d^2, \text{ etc.}; \Sigma ac\cos\hat{ac} = 0.$$

---

\* Catalan's *Théorèmes et Problèmes de Géométrie Élémentaire*, pp. 53, 54 (1879).

5.  $4(\text{area of quadrilateral}) = (b^2 + d^2 - c^2 - a^2) \tan \widehat{\delta\delta'}$ .

5a. Hence, or otherwise construct a quadrilateral, having given its four sides and area.

6. To find the cosine of the angle between any pair of opposite connectors.

[Equate the values of  $\lambda^2$ ,  $\mu^2$ ,  $\nu^2$  in 3° with those of (1), (2), (3).

7. If any point  $D$  be joined to the vertices of a triangle  $ABC$ ; the area of the triangle formed by joining the orthocentres of  $BCD$ ,  $CDA$ ,  $DAB$  is equal to  $ABC$ .

[Let  $O_1$ ,  $O_2$ ,  $O_3$  denote the orthocentres.  $DO_1$ ,  $DO_2$ ,  $DO_3$  are equal to  $a \cot A$ ,  $b \cot B$ ,  $c \cot C$  respectively, and are mutually inclined at angles  $A$ ,  $B$ ,  $C$ ; therefore, etc.]

8. If the vertices of a quadrilateral  $ABCD$  be joined to the orthocentres  $O$ ,  $O_1$ ,  $O_2$ ,  $O_3$  of the four triangles formed by their vertices taken in triads; to prove that

$$O \cdot ABCD = O_1 \cdot ABCD = O_2 \cdot ABCD = O_3 \cdot ABCD.$$

[Let the angle  $AOD = \theta$ . Taking any of the anharmonic ratios of the pencil  $O \cdot ABCD$  and reducing, we obtain

$$\frac{\sin \theta \sin B}{\sin(B + \theta) \sin A} = \frac{b}{a} \frac{\sin \theta}{\sin(B + \theta)} = \frac{bd \sin OAD}{ac \sin OCD} = \frac{bd \cos \widehat{bd}}{ac \cos \widehat{ac}} = \frac{\nu^2 - \lambda^2}{\nu^2 - \mu^2}$$

(Ex. 4, 4°). It follows generally that the six anharmonic ratios of the pencil  $O \cdot ABCD$  are  $\frac{\lambda^2 - \mu^2}{\lambda^2 - \nu^2}$ ,  $\frac{\mu^2 - \nu^2}{\mu^2 - \lambda^2}$ ,  $\frac{\nu^2 - \lambda^2}{\nu^2 - \mu^2}$  and their reciprocals. Similarly for the remaining pencils  $O_1 \cdot ABCD$ , etc. Russell.]

#### NOTE ON PASCAL AND BRIANCHON'S HEXAGONS.

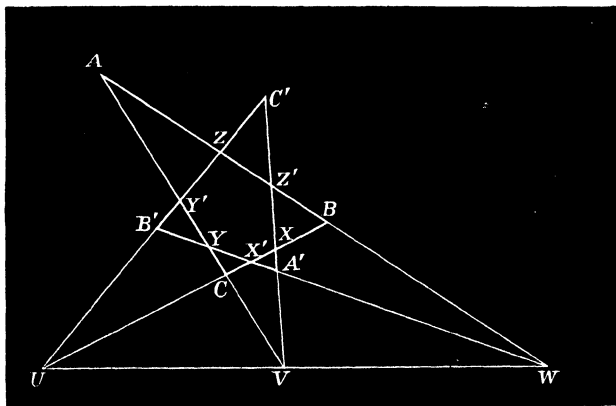
When two triangles  $ABC$  and  $A'B'C'$  are in perspective, the lines  $AA'$ ,  $BB'$ ,  $CC'$  are concurrent; therefore  $A$  and  $A'$ ,  $B$  and  $B'$ ,  $C$  and  $C'$  may be regarded as the opposite vertices of a Brianchon hexagon, and the centre of perspective of the two triangles is the Brianchon point of the hexagon.

But in this case we have three other pairs of triangles in perspective, viz.,  $BCA'$  and  $B'C'A$ ,  $CAB$  and  $C'A'B$ ,  $ABC$  and  $A'B'C$ . Hence with the vertices of two triangles in perspective we can form four Brianchon hexagons having the same Brianchon

point, the opposite vertices of the hexagons being in each case corresponding vertices of the two triangles.

Again, if the non-corresponding sides of the triangles intersect as in figure in points  $X$  and  $X'$ ,  $Y$  and  $Y'$ ,  $Z$  and  $Z'$ , and the corresponding sides in  $UVW$ ,  $UVW$  is the axis of perspective.

But in this case we have three other pairs of triangles in perspective to the same axis, viz., those obtained by interchanging a pair of corresponding sides, e.g., if  $L, M, N$  and  $L', M', N'$  denote the sides of the given triangles, it is obvious that the triangles  $LMN'$  and  $L'M'N$ ,  $MNL'$  and  $M'N'L$ ,  $NLM'$  and  $N'L'M$  have the same axis of perspective; hence with the sides of two triangles in perspective we may form four Pascal hexagons having a common Pascal line, i.e., the axis of perspective of the triangles, the corresponding sides of the triangles being in each case opposite sides of the hexagons.



In the accompanying figure the legs of the angles whose vertices are at  $U, V$  and  $W$  intersect again in twelve points, viz.,

$$X, X', Y, Y', Z, Z', A, A', B, B', C, C',$$

and these we have seen may be grouped in four different ways into two groups of six ( $XX'YY'ZZ'$ ), and  $AA'BB'CC'$  determining Pascal and Brianchon hexagons respectively; also that the alternate sides



$XX'$  and  $YY'$ ) of the Pascal hexagon intersect (in  $C$ ) in six points, which form a Brianchon hexagon.

Again, since sixty Pascal hexagons may be formed from the points  $XX'YY'ZZ'$ , and  $YY'$  and  $ZZ'$  meet in  $A$ , and  $YX'$  and  $Z'X$  in  $A'$ , by taking these lines as pairs of opposite sides of one of the hexagons ( $YY'XZ'ZX'$ ),  $AA'$  is its Pascal line; similarly  $BB'$  and  $CC'$  are Pascal lines of the hexagons  $XX'YZ'ZY'$  and  $XX'ZY'YZ'$  respectively; but  $AA'$ ,  $BB'$  and  $CC'$  are concurrent, hence *the sixty Pascal lines pass in threes through twenty points.*

Similarly it may be shewn that of the sixty *Brianchon hexagons* formed by the conjugate hexad of points  $AA'BB'CC'$ , their Brianchon points lie in triads on twenty lines. And either property involves the other as will be seen by reciprocation with respect to a circle.

## CHAPTER VI.

### INVERSE POINTS WITH RESPECT TO A CIRCLE.

**Def.** Two points  $P$  and  $Q$  are inverse with respect to a circle when the line  $PQ$  passes through the centre  $O$  and  $OP \cdot OQ =$  the square of the radius of the circle.

For the circle of unit radius  $OP \cdot OQ = 1$  or  $OP$  is the inverse, or reciprocal, of  $OQ$ .

69. It appears from the definition (1°) That inverse points are in the same direction from the centre when the circle is real and in opposite directions when the radius is imaginary, that is when it is of the form  $R\sqrt{-1}$ . (2°) They coincide on the circle; and when the radius is not real the inverse  $Q$  of a point  $P$  at a distance  $OP$  from the centre is given by the equation  $OP \cdot OQ = -R^2$ . (3°) When either coincides with the centre the other is at infinity.

**70. Theorem.**—*If a line  $AB$  be divided internally and externally in  $P$  and  $Q$  in the same ratio,  $P$  and  $Q$  are inverse points with respect to the circle on  $AB$  as diameter; also  $A$  and  $B$  are inverse points with respect to the circle on  $PQ$ .*

For if  $M$  be the middle point of  $AB$ , by hyp.,

$$\frac{AP}{BP} = \frac{AQ}{BQ}, \text{ hence } \frac{AM + MP}{BM - MP} = \frac{AM + MQ}{QM - MB'}$$

by taking the sum to difference on each side we have

$$\frac{AM+BM}{2MP} = \frac{2MQ}{AM+BM}; \text{ therefore } MP \cdot MQ = MA^2.$$

A similar proof applies to show that

$$NA \cdot NB = NP^2 = NA^2,$$

where  $N$  is the middle point of  $PQ$

71. Since  $MP \cdot MQ = MN^2 - PN^2$ , (Euc. II. 6)  
therefore (Art. 70)  $AM^2 = MN^2 - PN^2$ ,  
or transposing,  $MN^2 = AM^2 + PN^2$ .

Hence for any two segments  $AB$  and  $PQ$  placed to divide each other harmonically, the square of the distance ( $MN$ ) between their middle points = the sum of the squares of half the segments.

#### EXAMPLES.

1. The distances of the points of contact of the in- and ex-circles of a triangle with the sides measured from any vertex on either of the sides passing through it are  $s$ ,  $s-a$ ,  $s-b$ ,  $s-c$ .

2. If  $M$  denote the middle point of the base ( $c$ ) of a triangle,  $Q$  the intersection with the base of the fourth common tangent to the ex-circles  $O_1$  and  $O_2$ ,  $P$  the foot of the perpendicular from the vertex on the base,  $MP \cdot MQ = \left(\frac{a+b}{2}\right)^2$ .

[For  $O_1O_2$  is divided harmonically in  $C$  and  $Q$ , project  $O_1$ ,  $O_2$ , and  $C$  on base and apply Art. 70].

3. Show also that the rectangle under the distances of the middle point of the base from the feet of the perpendicular and internal bisector of vertical angle = square on half the difference of sides.

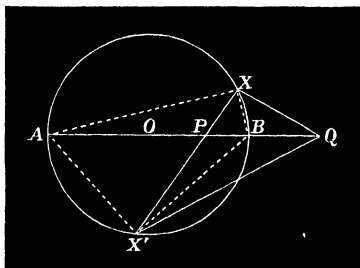
72. **Theorem.**—The distances of any point  $X$  on a circle from a pair of inverse points have a constant ratio.

Since  $OQ : OX = OX : OP$ ; the two triangles  $OQX$  and  $OX P$  are similar (Euc. VI. 6),

and (Euc. VI. 4)  $\frac{PX^2}{QX^2} = \frac{PO^2}{OX^2} = \frac{OX^2}{OQ^2}$ ;

therefore  $\frac{PX^2}{QX^2} = \frac{OP}{OQ}$ ,

or the squares of the distances of a variable point ( $X$ ) on a circle from a pair of inverse points ( $P, Q$ ) are as the distances of these points from the centre.



COR. 1. Let  $X$  coincide with each extremity of the diameter  $AB$  containing the points, then

$$\frac{PX^2}{QX^2} = \frac{PA^2}{QA^2} = \frac{PB^2}{QB^2} = \frac{OP}{OQ}.$$

COR. 2. Given a triangle ( $PQX$ ), the base ( $PQ$ ), and ratio of sides, the locus of the vertex is a circle ( $ABX$ ) with respect to which the extremities of the base are inverse points.

COR. 3. If the ratio of sides in Cor. 2 = 1, the locus is a line bisecting the base at right angles, therefore *the reflexion of a point is its inverse with respect to the line*.

COR. 4. From Cor. 1.  $AX$  and  $BX$  are the bisectors of the angle  $PXQ$ .

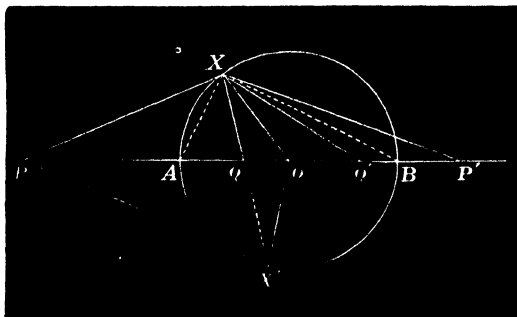
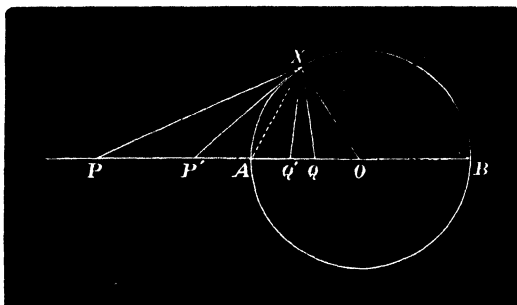
COR. 5. If  $PX$  be produced to meet the circle again in  $X'$ ,  $A$  and  $B$  are the centres of the in- and ex-circles of the triangle  $QXX'$ . (By Cor. 4.)

COR. 6. The line  $PQ$  containing a pair of inverse points bisects the angle  $(XQX')$  which any chord through either  $(P)$  subtends at the other.

COR. 7. The quadrilateral  $OQXX'$  is cyclic.

[For  $OXX' = PQX$ , but  $OXX' = OX'X$ ; therefore, etc. Euc. III. 21.]

COR. 8. For any other pair of inverse points  $P', Q'$  on the diameter  $AB$ ; the angles  $PXP'$  and  $QXQ'$  are equal or supplemental according as the pairs of points are taken in the same or opposite directions from the centre.



[The angles  $PXQ$  and  $P'XQ'$  have in either case the same bisectors  $AX$  and  $BX$ .]

EXAMPLES.

1. Any circle passing through a pair of inverse points  $P$  and  $Q$  with respect to a given one cuts the latter orthogonally.

[From the definition of inverse points and *Eucl. III. 37.*]

2. To find two points  $P$  and  $Q$  which shall be inverse with respect to two given circles.

[The circle passing through any point and its inverses with respect to each of the given circles meets their line of centres in the points required.]

3. The line  $L$  bisecting  $PQ$  at right angles is such that the tangents from any point  $O$  on it to either of the circles in *Ex. 2* are equal to  $OP$  or  $OQ$ .

[For the circle with  $O$  as centre and  $OP=OQ$  as radius meets the given circles orthogonally ; \* therefore, etc.]

4. Any two pairs of inverse points are concyclic.

5. Any chord  $XY$  of a circle passing through  $P$  is divided harmonically by  $P$  and the perpendicular to  $PQ$  through  $Q$ .

[For the angle  $XQY$  is bisected internally and externally by the lines at right angles.]

6. The radical axes  $L, M, N$  of three circles taken in pairs are concurrent.

[For the point  $(L, M)$  of intersection of any two is the centre of the circle cutting the three given ones orthogonally.]

**Def.** This point of concurrence  $O$  is the *Radical Centre* of the circles, and is such that for any three secants  $XX', YY', ZZ'$  drawn through it to the circles respectively

$$OX \cdot OX' = OY \cdot OY' = OZ \cdot OZ'.$$

The common value of these rectangles is called the *Radical Product* of the circles, and is equal to the square of the tangents to them when  $O$  is outside the circles. (*See Art. 23, Ex. 11, footnote.*)

\* Hence the locus of a point from which tangents to two circles are equal is a right line, viz., the axis of reflexion of their common pair of inverse points. It is termed the *Radical Axis* of the circles, and is their chord of intersection, *real or imaginary*.

7. The radical axis of two intersecting circles is their chord of intersection ; hence show that the common chords of three circles taken in pairs are concurrent.

8. Describe a circle meeting three given circles at right angles.

9. For any triangle  $ABC$  find a point  $O$  such that

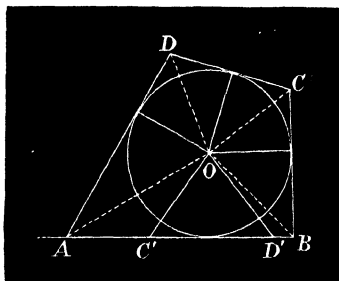
$$OA : OB : OC = \text{given ratios.}$$

10. For any four collinear points  $A, B, C, D$  find the loci of points ( $1^\circ$ ) such that the angles  $AOB$  and  $COD$  are equal, ( $2^\circ$ )  $BOC$  is supplement of  $AOD$ .

11. For any six collinear points taken in the order  $ABCC'B'A'$  find  $O$  such that the angles  $BOC, COA, AOB$  are respectively equal to  $B'OC', C'OA', A'OB'$ .

[By Ex. 10.]

12. The four sides of an escribed quadrilateral  $ABCD$  being given in magnitude and  $AB$  in position ; find the locus of the centre  $O$  of the in-circle.

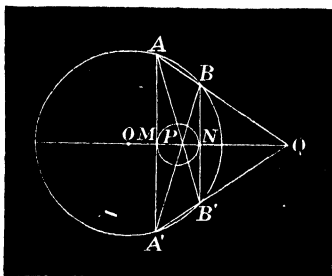


[Make  $AD' = AD$  and  $BC' = BC$ . Since  $OA, OB, OC, OD$  are the bisectors of the angles of the quadrilateral, it is easy to see that  $\angle AOB + \angle COD = \pi$ . Again the triangles  $AOB$  and  $AOD'$  are equal in every respect (Euc. I. 4) ; hence  $\angle ADO = \angle AD'O$  ; similarly  $\angle BCO = \angle BC'O$  ; therefore by addition it follows that  $\angle C'OD' = \angle COD$  or  $\angle AOB + \angle C'OD' = \pi$ , and the required locus is a circle having  $A, B$  and  $C', D'$  pairs of inverse points. Dilworth.]

13. The centres of perspective  $P$  and  $Q$  of any two parallel chords  $AA'$  and  $BB'$  of a circle are inverse points with respect to the circle, and the circle touching the chords at their middle points.

[For we have  $PA=PA'$ ,  $PB=PB'$ ,  $QA=QA'$  and  $QB=QB'$ ; hence  $PA/QA=PB/QB$ =etc.; therefore, etc.

The second part follows, since  $MN$  is divided harmonically by  $P$  and  $Q$ . Art. 70.]



13a. To what does the theorem reduce when  $AA'$  and  $BB'$  coincide?

14. For any two pairs of inverse points  $P, Q$  and  $P', Q'$  prove that

$$\frac{PP' \cdot PQ'}{QP' \cdot QQ'} = \frac{OP}{OQ'} \quad \left( = \frac{PA^2}{QA^2} = \frac{PB^2}{QB^2} \right).$$

[ $PP'QQ'$  is a cyclic quadrilateral (Ex. 4); hence the triangles  $OPP'$  and  $OQ'Q$  are similar; so also are  $OP'Q'$  and  $OQ'Q$ ; therefore, etc. (Euc. VI. 4). Otherwise if  $p$  and  $q$  denote the perpendiculars from  $P$  and  $Q$  on  $OP'Q'$ , we have

$$PP' \cdot PQ' = p \cdot D, \text{ and } QP' \cdot QQ' = q \cdot D;$$

hence 
$$\frac{PP' \cdot PQ'}{QP' \cdot QQ'} = \frac{p}{q} = \frac{OP}{OQ'}.]$$

15. If  $P, Q, R$  be any three collinear points on the diagonal triangle of a quadrilateral; their harmonic conjugates  $P'Q'R'$  with respect to the diagonals  $XX', YY', ZZ'$  are also collinear.

[For  $XX'$  is divided harmonically in  $B$  and  $C$  (Art. 68) and  $P$  and  $P'$ ; hence, by Ex. 14,

$$\frac{BP \cdot BP'}{CP \cdot CP'} = \frac{BX^2}{CX^2} = \frac{BL}{CL} \quad (\text{where } LX=LX').$$

Similarly 
$$\frac{CQ \cdot CQ'}{AQ \cdot AQ'} = \frac{CY^2}{AY^2} = \frac{CM}{AM} \quad (\text{where } MY=MY'): \text{ etc.}$$



Multiplying, we have

$$\frac{BP}{CP} \cdot \frac{CQ}{AQ} \cdot \frac{AR}{BR} \cdot \frac{BP'}{CP'} \cdot \frac{CQ'}{AQ'} \cdot \frac{AR'}{BR'} = 1 = \frac{BL}{CL} \cdot \frac{CM}{AM} \cdot \frac{AN}{BN}, *$$

but  $P, Q, R$  are in a line ; † therefore, etc.]

16. To what does Ex. 15 reduce when the line  $PQR$  is at infinity ?

17. The angles subtended by the diagonals of a complete quadrilateral at any point  $O$  have a common angle of harmonic section, real or imaginary.

[ $O$  is the point of intersection of the lines  $PQR$  and  $P'Q'R'$  in Ex. 16 ; therefore, etc.]

18. The circles on the diagonals of a complete quadrilateral pass through two points, real or imaginary.

[In Ex. 17, if two of the angles  $NOX'$ ,  $NOY'$  are right ;  $NOZ'$  must also be right, ‡ since it is divided harmonically by  $PQR$  and  $P'Q'R'$ .]

19. Any transversal to the pencil in Ex. 17 is cut in six points which, taken in pairs, have a common segment of harmonic section.

20. To what does Ex. 17 reduce when  $O$  is at infinity ?

21. If the sides of a triangle  $ABC$  are divided harmonically in  $XX'$ ,  $YY'$ ,  $ZZ'$  ; if  $X, Y, Z$  are collinear, the middle points  $L, M, N$  of these segments are collinear.

22. If perpendiculars be let fall on the sides of a triangle from a pair of inverse points  $O$  and  $O'$  and their feet joined ; the triangles  $PQR$  and  $P'Q'R'$  thus formed are similar and their areas are as the distances of  $O$  and  $O'$  from the circum-centre.

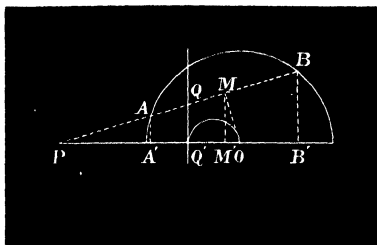
[For	$QR = AO \sin A$ , and $Q'R' = AO' \sin A$ ,	
therefore	$QR/Q'R' = AO/AO'$ ;	
similarly	$RP/R'P' = BO/BO'$ , etc.	Art. 72.]

\* Hence also the middle points  $L, M, N$  of the diagonals of a complete quadrilateral are collinear.

†  $PQR$  and  $P'Q'R'$  are termed *Conjugate Lines* of the quadrilateral.

‡ Generally, For a number of angles at a common vertex having a common angle of harmonic section if any two are right, all the others are also right.

23. Through a point  $P$  in the diameter of a semi-circle draw a chord  $AB$  such that the area of the quadrilateral  $ABA'B'$ , where  $A'B'$  is the projection of  $AB$  on the diameter, may be a maximum.



[Let  $Q'$  be the inverse of  $P$  with respect to the circle; draw  $QQ'$  at right angles to  $A'B'$ . Project  $M$  the middle point of  $AB$  on  $A'B'$  and let  $X$  be the intersection of  $MM'$  with the semi-circle on  $Q'O$ . Then area  $S$  of quadrilateral  $ABA'B' = A'B' \cdot MM'$ , hence

$$\begin{aligned} S^2 &= 4MM'^2 \cdot A'B'^2 = 4OM' \cdot PM' \cdot M'P \cdot M'Q', \text{ by Art 70,} \\ &= 4PM'^2 \cdot OM' \cdot M'Q' = 4PM'^2 \cdot M'X^2; \end{aligned}$$

or  $S$  is equal to the area of the maximum rectangle that can be inscribed in a given circle, one of whose sides is parallel to a given line. Art. 14, Ex. 2.]

24. Six perpendiculars are drawn from the inverse of the intersection of the diagonals of a cyclic quadrilateral to the sides and diagonals. Show

1°. The feet of those to the sides are collinear.

2°. The line of collinearity bisects at right angles the line joining the feet of perpendiculars on the diagonals.

[By method of Ex. 22.]

25. If  $XX'$ ;  $YY'$ ;  $ZZ'$  denote the feet of the bisectors of the angles of a triangle  $ABC$ , show that the pedal triangles of two points  $O$  and  $O'$  inverse to any of the circles on these segments as diameters, with respect to  $ABC$ , are *inversely* similar. (Neuberg.)

[Let  $O$  and  $O'$  be inverse with respect to  $ZZ'C$ ,\*  $PQR$  and  $Q'P'R'$  their pedal triangles respectively.  $M$  the middle point of  $ZZ'$ .

Then 
$$\frac{PQ^2}{PQ'^2} = \frac{MO}{MO'},$$

and 
$$\frac{R'P}{R'Q'} = \frac{AO \sin A}{BO \sin B} = \frac{BO \sin B}{AO \sin A} \text{ (by Ex. 14) } = \frac{RP}{RQ};$$

also the angles  $R$  and  $R'$  are equal ; therefore etc.

NOTE.—If  $O$  is on the circle  $ZZ'C$  the pedal triangle is isosceles, similarly if it is the point of intersection of the circles  $ZZ'C$  and  $YY'B$  it is isosceles in a double aspect, *i.e. equilateral*.

Hence we may infer that *the circles  $AXX'$ ,  $BYY'$ , and  $CZZ'$  pass through two points  $O$  and  $O'$  which are inverse (Ex. 22) with respect to the circum-circle of  $ABC$  and whose pedal triangles with respect to  $ABC$  are equilateral.*]

\* *Le cercle d'Apollonius du triangle  $ABC$  par rapport à  $AB$ .* V. *Educ. Times*, Dec., 1890.

## CHAPTER VII.

### POLES AND POLARS WITH RESPECT TO A CIRCLE.

#### SECTION I.

#### CONJUGATE POINTS, POLAR CIRCLE.

**73. Def.** The perpendicular to the line joining a pair of inverse points passing through either is the *Polar* of the other with respect to the circle. In the figure of Art. 74  $C$  and  $Z$  are inverse points; and  $C$  and the line  $AB$  are termed *Pole and Polar* with respect to the circle.

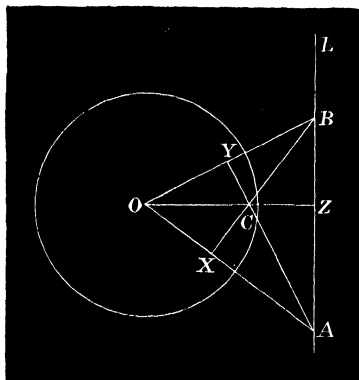
Any point  $A$  or  $B$  on the polar is the *Conjugate* of  $C$ , hence the polar of a point is the locus of its conjugates.

Again, since the circle on  $BC$  as diameter passes through  $Z$  and therefore cuts the given one orthogonally:—  
1°. *The circle described on the line joining two conjugate points cuts the given circle orthogonally.* 2°. *The distance between two conjugate points is equal to twice the length of the tangent to the circle from the middle point of the line connecting them.*

**74. Theorem.**—*For any two conjugate points  $B$  and  $C$ , to prove that each lies on the polar of the other with respect to the circle.*

Suppose the polar of  $C$  to be  $AB$ , we require to prove that the polar  $B$  passes through  $C$ . Join  $AO$ , draw a

perpendicular to it  $CX$ . Then evidently (Euc. III. 36)  $OA \cdot OX = OC \cdot OZ = r^2$ ; hence  $CX$  is the polar of  $A$ . Thus as the point  $B$  moves along the line  $AB$  its polar



(a)

turns around, or envelopes,  $C$ . At  $Z$  therefore the polar is the chord of contact of tangents through that point to the circle.

#### EXAMPLES.

1. The extremities of any diameter of a circle which cuts a given one orthogonally are conjugate points with respect to the latter.

2. If a variable chord  $AB$  of a circle pass through a fixed point  $P$ ; the locus of the intersection of tangents to the circle at  $A$  and  $B$  is a line.

[The polar of  $P$  with respect to the circle.]

3. The diameter  $AB$  of a circle is the polar of a point at infinity in a direction perpendicular to  $AB$ .

4. The locus of a point which has a common conjugate with respect to three circles is their common orthogonal circle.

**75. Theorem.**—If  $A$  and  $B$  be any two points and  $L$  and  $M$  their polars with respect to a circle, the point  $LM$  is the pole of the line  $AB$ .

For  $LM$  is conjugate to both  $A$  and  $B$ , hence the line joining  $A$  and  $B$  is its polar (Art. 73), or “the line of connexion of any two points is the polar of the point of intersection of the polars of the points.” Townsend.

**76.** More generally for three points  $A, B, C$  and their polars  $L, M, N$ , denoting the points  $MN, NL, LM$  by  $A', B', C'$  respectively; we see as above that  $A', B', C'$  are the poles of  $BC, CA, AB$ ; hence, for any two triangles if the vertices of either are the poles of the corresponding sides of the other; then, reciprocally, the vertices of the latter are the poles of the corresponding sides of the former.

**Def.** Such triangles are said to be *Reciprocal Polars* with respect to the circle.

**77.** In the particular case when  $ABC$  and  $A'B'C'$  coincide, the triangle is *Self-Reciprocal* with respect to the circle. It is manifest, since each vertex is the pole of the opposite side, every two of its vertices are conjugate points; and the triangle is therefore termed *Self-Conjugate* with respect to the circle.

Its centre  $O$  coincides with the orthocentre  $O$  of  $ABC$  and the square of its radius ( $\rho$ ) is given by

$$\rho^2 = OA \cdot OX = OB \cdot OY = OC \cdot OZ,$$

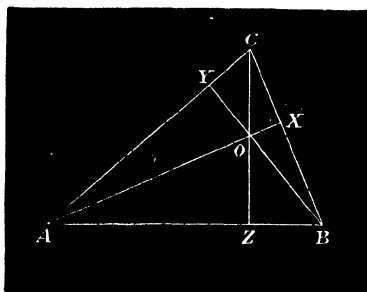
where  $X, Y, Z$  are the feet of the perpendiculars of the triangle.

This circle is called the *Polar Circle* of the triangle.

**NOTE.**—In order that the polar circle may be real, the pairs of points  $A$  and  $X, B$  and  $Y, C$  and  $Z$ , which are inverse with respect to it, must lie in the same direction from its centre  $O$ . It is therefore

real when the triangle is obtuse angled, and imaginary for acute angled triangles.

78. *Expressions for the Radius ( $\rho$ ) of the Polar Circle.*



(β)

Let  $O$  be the ortho-centre of  $ABC$ , then it appears that  $A, B, C$  are the ortho-centres of  $BOC, COA$ , and  $AOB$  respectively. For this reason the four points  $A, B, C, O$  are said to form an *Orthocentric System*.

Also the circum-circles of the four triangles  $BOC, COA, AOB$ , and  $ABC$  are equal.

Hence since  $a$  and  $AO$ , chords of equal circles, subtend complementary angles at the circumferences,

$$a^2 + AO^2 = b^2 + BO^2 = c^2 + CO^2 = d^2, \dots \dots \dots (1)$$

also (fig. β)  $a^2 = BO^2 + CO^2 + 2CO \cdot OZ$ , (Euc. II. 13)  
therefore by substitution from (1)

$$a^2 = 2d^2 - b^2 - c^2 + 2CO \cdot OZ,$$

or  $-CO \cdot OZ = d^2 - \frac{1}{2}(a^2 + b^2 + c^2) = \rho^2 \dots \dots \dots (2)$

This formula is equivalent to  $\rho^2 = d^2 \cos A \cos B \cos C$ , by reduction or independently, as follows:—

$$-\rho^2 = OC \cdot OZ = OC \cdot \frac{OA \cdot OB}{d} = d^2 \cos A \cos B \cos C, \dots (3)$$

since a chord is equal to the diameter of the circle into the sine of the angle it subtends.

EXAMPLES.

1. The four polar circles of the triangles  $BOC$ ,  $COA$ ,  $AOB$  and  $ABC$  are mutually orthogonal.\*

[Let their radii be  $\rho_a$ ,  $\rho_b$ ,  $\rho_c$ ,  $\rho$ . Since their centres are at  $A$ ,  $B$ ,  $C$ ,  $O$ , by *Euc. II. 2*,

$$AB^2 = AB \cdot AZ + AB \cdot BZ = \rho_a^2 + \rho_b^2;$$

therefore, etc.]

2.  $B$  and  $C$ ,  $C$  and  $A$ ,  $A$  and  $B$  are pairs of conjugate points with respect to the polar circles of  $BOC$ ,  $COA$  and  $AOB$  respectively.

3. The square of the distance  $BC$  between any two conjugate points is equal to the sum of the squares of the tangents drawn from them to the circle.

[By *Ex. 1* the tangents from  $B$  and  $C$  to the circle  $\rho_a$  are the radii of  $\rho_b$  and  $\rho_c$ , but  $BC^2 = \rho_b^2 + \rho_c^2$ ; therefore, etc.]

4. Prove that  $AZ \cdot BZ = t^2$ , where  $t$  is the tangent to the polar circle from  $Z$ , the *Polar Centre*† of  $AB$ ; and conversely.

[By similar triangles  $ACZ$  and  $OBZ$ ,  $AC : CZ = OZ : BZ$ , etc.]

5. Conjugate points  $A$  and  $B$  with respect to any chord  $MN$  are conjugate with respect to the circle.

[For the polar centre  $Z$  of  $AB$  is the middle point of  $MN$ ; but (hyp.)  $ZA \cdot ZB = ZM^2 = -ZM \cdot ZN$  or the square of the imaginary tangent from  $Z$  to the circle; therefore, etc., by *Ex. 4*.]

6. If a number of circles have a common orthogonal circle, the extremities of any diameter of the latter are conjugates with respect to the entire system.

7. On a given line find two points which shall be conjugates to each of two given circles.

[The middle point of the required segment is such that the tangents from it to the circles are equal; therefore, etc., by *Art. 72*, *Ex. 3*.]

\* Hence :—If four circles are mutually orthogonal, their centres form an orthocentric system and one of the circles is imaginary.

†  $Z$  the foot of the perpendicular from the centre on  $AB$  is also called the *Middle Point* of the line. (Cf. *Euc. III. 3*.)



8. On a given circle  $O$  find two points  $A$  and  $B$  which shall be conjugates to each of the circles  $C, r_1$ ;  $D, r_2$ .

[The middle point  $M$  of the required chord is on the radical axis  $L$  of the given circles (Art. 72, Ex. 3). Let  $2t$  be the length of  $AB$ ; then  $CM^2 = t^2 + r_1^2 = r_1^2 + AM^2 = r_1^2 + r_2^2 - OM^2$ ; hence  $CM^2 + OM^2$  is known, and the triangle  $COM$  is completely determined; therefore, etc.]

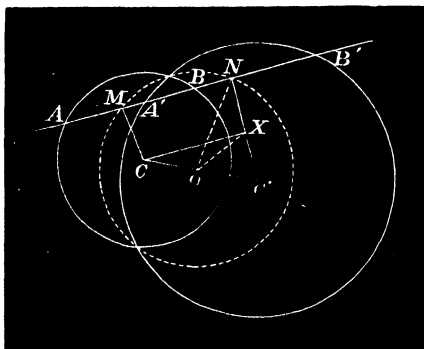
9. Place a chord of given length in a circle so that its extremities may be conjugates with respect to another.

[See Ex. 8.]

10. If a right line  $AB$  meet either  $(C, r)$  of two circles in conjugate points  $(A, B)$  with respect to the other; then reciprocally it meets the latter  $(C', r')$  in conjugate points  $(A', B')$  with respect to the former.

[For by Ex. 5  $AB$  divides  $A'B'$  harmonically, hence  $A'B'$  divides  $AB$  harmonically; therefore, etc.]

11. Find the locus of the middle points  $M$  and  $N$  of the chords  $AB$  and  $A'B'$  in Ex. 10.



$$\begin{aligned}
 [CM^2 + C'M^2 &= CN^2 + C'N^2 = CM^2 + C'N^2 + MN^2 \\
 &= CM^2 + C'N^2 + MB^2 + A'N^2 && \text{(Art. 71.)} \\
 &= r^2 + r'^2 = \text{const. ;}
 \end{aligned}$$

hence the required locus is a circle whose centre  $O$  is at the middle point of  $CC'$  and the square of whose radius is equal to  $\frac{1}{2}(r^2 + r'^2) - \delta^2$ ,

where  $2\delta = CC'$ . It evidently passes through the intersections of the given circles.]

12. Show that  $CM \cdot C'N = \text{const.}$

[Draw  $CX$  at right angles to  $C'N$ . Join  $OX$ . Since  $OC'X$  is an isosceles triangle and  $N$  a point in the base produced,

$$\begin{aligned} CM \cdot C'N &= C'N \cdot NX = ON^2 - OX^2 = ON^2 - OC'^2 \\ &= \frac{1}{2}(r^2 + r'^2 - 4\delta^2) = rr' \cos \theta, \end{aligned}$$

where  $\theta$  is the angle between the given circles ; therefore, etc.]

13. Any circle described around the polar centre of a triangle  $ABC$  meets the corresponding sides of the median triangle in  $A', B', C'$  such that  $AA' = BB' = CC'$ .

14. A tangent is drawn from the polar centre to the circum-circle, and from the point of contact a tangent is drawn to the polar circle, show that the angle between these lines is  $45^\circ$ .

15. Draw through  $P$  a line cutting each of two given circles in conjugate points with respect to the other.

[By Exs. 10 and 11.]

16. Draw a line cutting each of two circles  $X$  and  $Y$  in conjugate points with respect to a third ( $Z$ ).

[Let the required line meet  $Z$  in the points  $A$  and  $B$ . The middle point  $M$  of  $AB$  is the intersection of two known circles passing through the intersections of  $Z$  and  $X$  and  $Z$  and  $Y$  (Ex. 11), and is thus determined ; therefore, etc.]

## SECTION II.

**79. Salmon's Theorem.**—*The distances of any two points  $A$  and  $B$  from the centre  $O$  of a circle are proportional to the distances  $AM$  and  $BL$  of each from the polar of the other.*

Draw  $AB'$  and  $BA'$  perpendiculars to  $OB$  and  $OA$  respectively.



by multiplication of ratios, we have

$$\frac{OA^n}{AM \cdot AM_1 \cdot AM_2 \dots} = \frac{OB \cdot OB_1 \cdot OB_2 \dots}{BL \cdot B_1L \cdot B_2L \dots};$$

or, the product of the distances of a point ( $A$ ) from any number of lines ( $M$ ) is to the product of the distances of their poles ( $B$ ) from the polar ( $L$ ) of the point as the  $n^{\text{th}}$  power of the distance of the point from the centre is to the product of the distances of the poles from the centre.

COR. 4. If  $M, M_1, M_2$  in Cor. 3 form an inscribed polygon,  $B, B_1, B_2, \dots$  are the vertices of the corresponding escribed one; hence the product of the distances of any point from the sides of an in-polygon is to the product of the distances of the vertices of the corresponding ex-polygon from the polar of the point as the  $n^{\text{th}}$  power of the distance of the latter from the centre is to the product of the distances of the vertices of the ex-polygon from the centre.

COR. 5. The rectangle under the distances of the extremities of any chord from a tangent is equal to the square of the distance of its point of contact from the chord.

#### EXAMPLES.

1. The opposite vertices of an escribed quadrilateral are  $AA', BB', CC'$ ; to prove that

$OA \cdot OA' : OB \cdot OB' : OC \cdot OC' = AX \cdot A'X : BX \cdot B'X : CX \cdot C'X$ ,  
where  $X$  is a tangent to the circle at any point  $P$ .

[Let the corresponding pairs of sides of the in-quadrilateral be  $L, L'; M, M'; N, N'$ ; then since

$$\frac{OA}{AX} = \frac{OP}{PL} \text{ and } \frac{OA'}{A'X} = \frac{OP}{PL'}$$

multiplying these equations,  $\frac{OA \cdot OA'}{AX \cdot A'X} = \frac{OP^2}{PL \cdot PL'}$ ;

but  $PL \cdot PL' = PM \cdot PM' = PN \cdot PN'$ ; therefore, etc.]

2. If  $\alpha, \beta, \gamma$  denote the perpendiculars from any point on the circum-circle on the sides of an in-triangle,

$$\beta\gamma \sin A + \gamma\alpha \sin B + \alpha\beta \sin C = 0$$

or 
$$\frac{\alpha}{a} + \frac{\beta}{b} + \frac{\gamma}{c} = 0.$$

3. If  $\lambda, \mu, \nu$  be the perpendiculars from the vertices of any triangle upon a variable tangent to the in-circle,

$$\frac{\cot \frac{1}{2}A}{\lambda} + \frac{\cot \frac{1}{2}B}{\mu} + \frac{\cot \frac{1}{2}C}{\nu} = 0.$$

[Let  $A', B', C', P$  be the points of contact with the sides and any tangent, then  $\frac{OA}{\lambda} = \frac{r}{\alpha'}$ , where  $\alpha'$  is the perpendicular from  $P$  on  $B'C'$ .

Hence 
$$\Sigma \frac{OA \cdot B'C'}{\lambda} = r \Sigma \frac{B'C'}{\alpha'} = 0; * \quad (\text{Ex. 2})$$

but  $OA \cdot B'C' = 2r^2 \cot \frac{1}{2}A$ ; substituting, we have

$$\Sigma \cot \frac{1}{2}A / \lambda = 0.$$

A particular case of this has been noticed in Art. 55, Ex. 8.]

4. If the perpendiculars from the vertices on any tangent to the circum-circle of a triangle be  $\lambda, \mu, \nu$ ; to prove that

$$a\sqrt{\lambda} + b\sqrt{\mu} + c\sqrt{\nu} = 0.$$

[If  $P$  be the point of contact of the tangent to the circle, by Ptolemy's Theorem,

$$a \cdot AP + b \cdot BP + c \cdot CP = 0,$$

but  $AP^2 = 2r\lambda$ , etc., hence  $\Sigma a\sqrt{\lambda} = 0$ .]

5. For any point  $P$  on the in-circle whose distances from the sides are  $\alpha, \beta, \gamma$ ; to prove that

$$\cos \frac{1}{2}A \sqrt{\alpha} + \cos \frac{1}{2}B \sqrt{\beta} + \cos \frac{1}{2}C \sqrt{\gamma} = 0.$$

[Let  $\lambda', \mu', \nu'$  be the distances of the points of contact  $A', B', C'$  of the sides of  $ABC$  from the tangent at  $P$ ;  $\alpha', \beta', \gamma'$  the distances of  $P$  from the sides of  $A'B'C'$ .

By Ex. 4, 
$$\Sigma \alpha' \sqrt{\lambda'} = 0 \text{ or } \Sigma \frac{\alpha'}{\sqrt{\mu' \nu'}} = 0,$$

but 
$$\sqrt{\mu' \nu'} = \alpha' = \sqrt{\beta \gamma}; \quad (\text{Art. 79, Cor. 5.})$$

\* The angles of  $A'B'C'$  are respectively  $90 - \frac{1}{2}A$ ,  $90 - \frac{1}{2}B$ ,  $90 - \frac{1}{2}C$ ; therefore 
$$\alpha' : \beta' : \gamma' = \cos \frac{1}{2}A : \cos \frac{1}{2}B : \cos \frac{1}{2}C.$$

hence, on substituting, since  $a' = 2r \cos \frac{1}{2}A$ ,

$$\Sigma a' \sqrt{\lambda'} = 0 = \cos \frac{1}{2}A \sqrt{a}, \text{ therefore, etc.}]$$

NOTE.—The equations in Exs. 2 and 5 are known in Analytical Geometry to be those of the circum- and in-circles respectively, the given triangle  $ABC$  being taken as the triangle of reference. The expressions in Exs. 3 and 4 are the *Tangential Equations of the In- and Circum-Circles*.

6. If two triangles  $ABC$ ,  $A'B'C'$  are reciprocal polars, they are in perspective.

[Let the perpendiculars from  $A'B'C'$  on the sides of  $ABC$  be  $p_1, p_2, p_3$ ;  $q_1, q_2, q_3$ ;  $r_1, r_2, r_3$  respectively; then, by Salmon's Theorem,

$$\frac{OB'}{OC} = \frac{q_3}{r_2}; \frac{OC'}{OA'} = \frac{r_1}{p_3}; \frac{OA'}{OB} = \frac{p_2}{q_1};$$

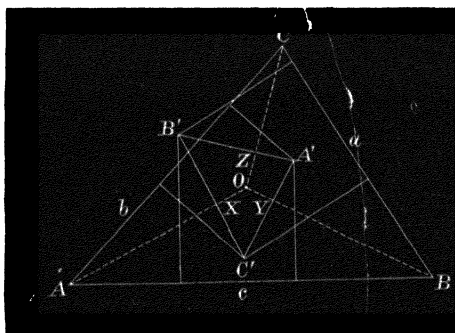
multiplying these equations we have

$$\frac{p_2}{p_3} \cdot \frac{q_3}{q_1} \cdot \frac{r_1}{r_2} = 1; \text{ therefore, etc. (Art. 65.)}]$$

7. A triangle inscribed in a circle is in perspective with the corresponding escribed one.

[By Ex. 6.]

8. Any two triangles may be so placed that the vertices of either are the poles of the sides of the other with respect to a circle.



[At the centre  $O$  of the required circle the sides of each triangle subtend angles *similar* to those of the other triangle. Find points

satisfying these conditions with respect to each triangle and place the latter with the points coincident and  $AO$  at right angles to  $B'C'$ ; then  $OB$  and  $OC$  will be at right angles to  $C'A'$  and  $A'B'$ . Again, since the perpendiculars from  $ABC$  on the sides of  $A'B'C'$  are concurrent, those from  $A'B'C'$  on the sides of  $ABC$  are also concurrent; it follows obviously that  $OA'$ ,  $OB'$ ,  $OC'$  are perpendicular to the sides of  $ABC$ ; and

$$OA \cdot OX = OB \cdot OY = \dots = OA' \cdot OX' = \dots = \rho^2.$$

9. To find the radius  $\rho$  of the circle in Ex. 8.

$$\left[ \frac{\text{area } B'OC'}{\text{area } ABC} = \frac{OB' \cdot OC'}{bc} = \frac{\rho^4}{bc \cdot OY' \cdot OZ'} = \frac{\rho^4}{4COA \cdot AOB} \right]$$

Similarly,  $\frac{C'OA'}{ABC} = \frac{\rho^4}{4} \cdot \frac{1}{AOB \cdot BOC}$ , etc.

Adding these results, we have

$$\frac{A'B'C'}{ABC} = \frac{\rho^4}{4} \cdot \sum \frac{1}{BOC \cdot COA} = \frac{\rho^4}{4} \cdot \frac{ABC}{BOC \cdot COA \cdot AOB},$$

or  $\rho^4 = \frac{4BOC \cdot COA \cdot AOB \cdot A'B'C'}{(ABC)^2} \quad \left. \right]$

10. The area of the reciprocal polar  $A'B'C'$  of a given triangle with respect to a circle is given by the equation of Ex. 9.

11. The minimum value of  $A'B'C'$  is obtained when the centre  $O$  coincides with the centroid of  $ABC$ ; and  $= \frac{27\rho^4}{4ABC}$

[In this case  $BOC = COA = AOB$ . Art. 14, Ex. 5.]

12. The reciprocal polar of the median triangle with respect to the in-circle or ex-circles of the given one is equal to  $ABC$ .

13. The reciprocal polar triangle may be of any species.

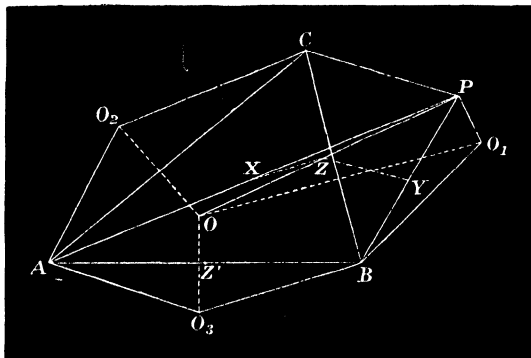
[Species depends on the position of the centre  $O$ .]

14. In Ex. 8  $O$  is one or other of two fixed points.

[One of them is obviously within both triangles and the sides of each subtend at it angles equal to the supplements of the angles of the other.

The other is the common intersection of the circles described *externally* on the sides of  $ABC$  containing angles equal to  $\pi - A'$ ,  $\pi - B'$ ,  $\pi - C'$ . On making the figure it will be observed that these circles, intersecting in pairs at the vertices of the triangle, can only

meet again in one point ; hence, if a point  $O$  be reflected with respect to the three sides of a triangle, the circles  $BCO_1$ ,  $CAO_2$ ,  $ABO_3$  meet in a point.\*



15. If the triangles  $ABC$  and  $A'B'C'$  are similar the second centre is any point on the circum-circle of  $ABC$  ; also if  $P$  be joined to  $A$ ,  $B$ , and  $O$  and  $X$ ,  $Y$ ,  $Z$  be the middle points of these lines and  $Z'$  the middle point of  $AB$  ;  $XYZZ'$  is a cyclic quadrilateral for

$$\angle XZY = \angle AOB \text{ and } \angle XZ'Y = \angle APB = \pi - \angle AOB ;$$

hence

$$\angle XZY + \angle XZ'Y = \pi ;$$

therefore  $Z$  the middle point of  $OP$  is on the nine-points-circle of  $ABP$ . Similarly it is on the nine-points-circles of the triangles with  $BC$  and  $AC$  as bases and  $P$  as vertex. Hence for any four points  $A$ ,  $B$ ,  $C$ ,  $P$ , the nine-points-circles of three of the triangles formed by them are concurrent. It is therefore obvious that all four *nine-points-circles* of the four triangles  $BCP$ ,  $CAP$ ,  $ABP$ ,  $ABC$  are concurrent.†

16. A triangle reciprocates into a similar one from either of the Brocard points as origin. (Art. 27.)

\* The points  $O$  and  $P$  are *reciprocally* related to the triangle  $ABC$ . For it will be seen that, if  $P$  be reflected with respect to the sides, the circles  $BCP_1$ ,  $CAP_2$ ,  $ABP_3$  will meet in  $O$ . It follows thence that the nine points circles of the triangles  $BCO$ ,  $CAO$  and  $ABO$  also pass through this point of concurrence.

† Van de Berg, *Mathesis*, t. 2, p. 141.



## SECTION III.

## RECIPROCATION.

80. If  $ABC \dots$  be any polygon and  $A'B'C' \dots$  another derived from it by taking the poles  $A', B', C', \dots$  of the sides  $BC, CA, AB$ , etc., with respect to any circle, then we have seen (Art. 76) that the vertices  $A, B, C$ , etc., of the former are the poles of the sides of the latter, and the two polygons are said to be *Reciprocal Polars* with respect to the circle. The process of deriving  $A'B'C' \dots$  is termed *Reciprocation*, and the circle, radius, and centre are the *Circle*, *Radius*, and *Centre*, or *Origin of Reciprocation*.

More generally, if  $ABC \dots$  be any curve to which tangents  $T_1, T_2, T_3, \dots$  are drawn at the points  $A, B, C, \dots$ , the locus of their poles is the *Reciprocal Polar Curve* of  $ABC \dots$  with respect to the circle. If the tangents at  $A$  and  $B$  are indefinitely near, their poles  $A', B'$  are also indefinitely near on the reciprocal curve; but the point  $T_1T_2$  is (Art. 76) the pole of the line  $A'B'$ ; hence in the limit the point  $A$  is the pole of the tangent at  $A'$ . The point  $A$  and tangent at  $A'$  are said to *correspond*. Thus, of two polar reciprocal curves any tangent to either corresponds to a point on the other, and each point of contact and the corresponding tangent are pole and polar with respect to the circle.

The following fundamental properties of two Reciprocal figures will appear obvious:—

1°. The line joining any two points of either is the

polar of the intersection of the corresponding lines of the other.

2°. Concurrent lines reciprocate into collinear points.

3°. The angle subtended by any two points of one at the origin is equal to the angle between the corresponding lines of the other.

4°. For any two figures  $X$  and  $Y$  and their reciprocals  $X'$  and  $Y'$ , the points of intersection of  $X$  and  $Y$  correspond to the common tangents to  $X'$  and  $Y'$ ; in other words, a common tangent to two curves corresponds to a point of intersection of their reciprocals.

5°. If  $X$  and  $Y$  touch, their reciprocals  $X'$  and  $Y'$  also touch, and each point of contact is the pole of the common tangent at the other.

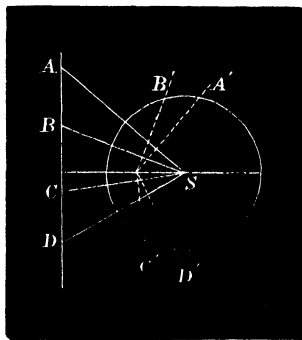
6°. Since two circles have four common tangents, real or imaginary, they reciprocate into curves which intersect in four points. (By 4°.)

7°. Any point connected with  $X$  and the tangents through it to the curve correspond to a line and its points of intersection with the reciprocal curve  $X'$ .

8°. The reciprocal of a circle is a *curve of the second degree*, i.e. one which meets every line in two points, real or imaginary. (By 7°.)

9°. The pencils determined by any four collinear points  $A, B, C, D$  at the origin  $S$  and the corresponding lines  $A', B', C', D'$  are similar.

[For the corresponding rays of pencils are at right angles.]



10°. Harmonic rows of points reciprocate into harmonic pencils of rays; and in the particular case when one point  $D$  of the row  $A, B, C, D$  coincides with the origin  $S$ ;  $SA', SB', SC'$  are in arithmetical progression.

11°. Parallel lines reciprocate into points collinear with the origin.

12°. A point and its polar reciprocate into a line and its pole with respect to the reciprocal curve. (Cf. 7°.)

### RECIPROCATION OF THE CIRCLE.

81. Let the origin  $S$  be outside the circle  $(O, r)$ ;  $OS = \delta$ ;  $L$  the polar of  $O$  with respect to the Circle of Reciprocation, and  $P$  the pole of any tangent to the circle at  $Z$ .

For the two points  $O$  and  $P$  we have, by Salmon's Theorem,

$$\frac{SP}{PL} = \frac{SO}{OZ} = \frac{\delta}{r} = \text{const.} = e \text{ (say).}$$

The locus of  $P$  given by the equation  $SP/PL = e$  is a *Conic Section*, of which  $S$  is termed a *Focus*,  $L$  a *Directrix*, and  $e$  the *Eccentricity*. (See Art. 79, Cor. 1.)

When  $e > 1$ , the conic is called a *Hyperbola*,

„  $e = 1$ , „ „ *Parabola*,

„  $e < 1$ , „ „ *Ellipse*.

Thus the reciprocal polar of a circle is a hyperbola, parabola, or ellipse, according as the origin is outside, upon, or within the circle.

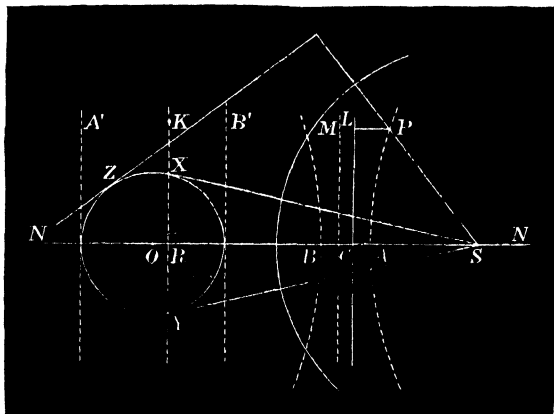
In the particular case when the origin coincides with the centre of the given circle, the reciprocal curve is a concentric circle.

Since the tangents to a circle are real and distinct from any points outside it, and reciprocate from  $S$  as

origin to two points at infinity; their points of contact  $X$  and  $Y$  reciprocate into two real tangents to the conic, meeting in  $C$  the correspondent of  $XY$ , whose points of contact are at infinity.

These lines are termed the *Asymptotes* of the hyperbola. They are *imaginary* for the ellipse, though they intersect in a real point, and *coincident* with the line at infinity for the parabola.

The tangents  $A'$  and  $B'$  at the extremities of the diameter  $OS$  correspond to points  $A$  and  $B$  called the foci of the conic; also since the distances of  $S$  from  $A, B'$ , are in H.P.,  $SA, SC, SB$  their reciprocals are in A.P. hence  $C$  is the middle point of the segment  $AB$ , and  $S$  is obviously the point at which the asymptotes intersect.\*



\* When the origin is outside the circle its polar divides the circumference into two parts which are respectively concave and convex to it.

These portions reciprocate into two distinct curves convex and concave to the origin as shown in the figure, and both branches reach to

Also since  $SA'$ ,  $SO$  and  $SB'$  are in A.P., their reciprocals  $SA$ ,  $SL$ ,  $SB$  respectively are in H.P.

The tangents from any point  $K$ , on  $XY$ , to the circle with  $XY$  and  $KS$  form an harmonic pencil (Art. 78, Ex. 5) hence by reciprocation any line through  $C$  meets the conic in an harmonic row of points, one of which, corresponding to the ray  $KS$ , is at infinity. Thus every chord of the conic through  $C$  is bisected. On account of this property  $C$  is termed the *Centre* of the curve.

Again, the tangents to the circle from any point on the perpendicular through  $S$  to  $RS$  and the lines that point to  $R$  and  $S$  form an harmonic pencil. By reciprocation any line parallel to  $OS$  meets the conic in an harmonic row of points, one of which, corresponding to the ray through  $S$ , is at infinity; another, corresponding to the ray through  $R$ , is on  $M$  the perpendicular through  $C$  to  $OS$ . It is therefore manifest that the conic is symmetrically situated with respect to this line. It is moreover symmetrical with respect to  $ON$ . These rectangular lines  $OM$ ,  $ON$  through the centre  $C$  are termed the *Axes* of the curve.

infinity. If, however, we assume in general that consecutive tangents to the circle reciprocate into consecutive points on the conic, by taking two tangents indefinitely near, one on the convex and the other on the concave part of the circle, we are led to the conclusions that the points at infinity on the opposite branches of the curves are indefinitely near, that the asymptotes are tangents at the points of coincidence, and that the hyperbola is a continuous curve.

EXAMPLES.

1. A circle, any point and its polar with respect to the circle, *e.g.*

Circle, centre and line at infinity.

Circle, origin and polar of origin.

Circle and inscribed polygon.

Circle (or conic) and self conjugate triangle.\*

A conic, a line and its pole with respect to the conic.

Conic, directrix and focus.

Conic, line at infinity and centre of conic.

Conic and escribed polygon.

Conic and self conjugate triangle.

2. The opposite sides of a cyclic hexagon meet in three collinear points. (Pascal.)

The opposite vertices of an escribed polygon connect by three concurrent lines. (Brianchon.)

This result follows when the circle described about the hexagon is taken as the circle of reciprocation.

In general, from any origin, the theorem of Pascal with respect to a circle reciprocates into Brianchon's property for a conic.

3. Four points on a circle subtend at a variable point on it equianharmonic pencils.

Four fixed tangents to a circle meet a variable tangent to it in equianharmonic rows ;

hence, generally from any origin, the property of *Euc. III. 21* becomes :—A variable tangent to a conic meets four fixed tangents in rows of points which are equianharmonic ; and reciprocally four fixed points on a conic subtend equianharmonic rows at a variable fifth point on it.

And again it follows conversely that, if two points connect equianharmonically with four others, all six lie on a conic ; hence :—Any two of the hexad of points connect equianharmonically with the remaining four. This system is sometimes called an *Equianharmonic Hexagon*. (Townsend, *Mod. Geom.* vol. II. p. 168.)

4. Concentric Circles.

Conics having same focus (origin) and directrix.

---

\* If the origin is taken at one of the vertices of the triangle the corresponding side of the reciprocal triangle is therefore at infinity, and its other two sides are diameters (*conjugate*) of the conic. See Exs. 8, 9.

5. Circles having a common pair of inverse points (from either point as origin).

Conics having a common focus and centre.

From the symmetry of the conic we infer that such a system has a second common focus ; hence :—*Coaxial Circles reciprocate from either of their common pair of inverse points into a system of Confocal Conics.*

6. Euc. III. 35, 36.

The rectangle under the distances of either focus from a pair of parallel tangents is constant ;

hence from symmetry we infer that the rectangle under the distances of the foci from any tangent is constant ; and conversely, the envelope of a variable line, the product of whose distances from two fixed points is constant, is a conic having the fixed points for foci.

7. A chord of a circle which subtends a right angle at the origin envelopes a conic.

The locus of the intersection of rectangular tangents to a conic is a circle.

(*Director Circle.*)

8. A variable chord of a circle passing through a fixed origin is divided harmonically by the point and its polar.

The variable chord of contact of two parallel tangents passes through and is bisected at the centre of the conic.

**Def.** The diameter of a conic parallel to a tangent is said to be *Conjugate* to that which passes through its point of contact.

9. Conjugate points with respect to a circle (from the pole of line joining them as origin).

Conjugate diameters of a conic.

10. If a variable point  $P$  moves on a line through the origin,  $S$  its polar passes through  $Q$  the pole of the line with respect to the circle ; and the tangents from  $P$  and the lines  $PQ$  and  $PS$  form an harmonic pencil.

If a variable chord of a conic moves parallel to a fixed direction, the harmonic conjugates of the points on it at infinity (*i.e.* the middle points) are collinear ;

hence *the locus of the middle points of any system of parallel chords is a line.*

11. Conjugate points coincide on the circle.

Each asymptote is its own conjugate.

12. The rectangle under their distances from the *middle* of the line joining them is constant.

The product of the tangents of the angles made by a pair of conjugate diameters with either axis of the conic is constant.

13. Euc. III. 21, 22.

The angles subtended at a focus by either pair of opposite sides of an escribed quadrilateral are equal or supplemental.

14. The locus of intersection of tangents containing a given angle is a concentric circle.

The envelope of a chord which subtends a constant angle at the focus is a conic having the same focus and directrix.

Their chord of contact envelopes a concentric circle.

The locus of the point of intersection of the tangents at the extremities is another conic having same focus and directrix.

15. If the vertex of an angle of given magnitude is on a circle, its variable chord of intersection envelopes a concentric circle.

If two points are taken on a fixed tangent so as to subtend a constant angle at the focus, the locus of the intersection of the tangents through them is a conic having same focus and directrix.

16. If the angle is right, the chord envelopes the centre (from vertex as origin).

The locus of intersection of rectangular tangents to a parabola is the directrix.

17. The perpendiculars of a triangle are concurrent.

The diagonals of a complete quadrilateral each subtend a right angle at a certain point ;

or the circles on the diagonals are concurrent.





It follows also, since any point (origin) on the circum-circle and the orthocentre are equidistant from the Simson line of the point, that *the locus of the orthocentre of a variable triangle escribed to a parabola is the directrix*

21. Having given base and vertical angle, the locus of the vertex of the triangle is a circle. (Euc. III. 21.)

If the extremities of a variable line, which subtends a constant angle at a fixed point, move on two fixed lines, it envelopes a conic to which these lines are tangents.

It therefore cuts them equianharmonically.

22. Since *inverse* segments subtend similar angles at any point on the circle, the segments of a line drawn across two circles subtend similar angles at either common inverse point.

The pairs of tangents to confocal conics from any point are equally inclined.

23. All circles meet in two imaginary points on the line at infinity.

Confocal conics have pairs of imaginary common tangents passing through the foci.

24. The polars of a point with respect to a system of coaxial circles are concurrent.

The poles of a line with respect to a system of confocal conics are collinear.

25. The two points in Ex. 24 are in perpendicular directions from either common inverse point.

The locus of the poles is a line perpendicular to the given one.

26. The sum of the squares of the segments of two rectangular chords of a circle is constant.

The sum of the squares of the reciprocals of the distances of the foci from two rectangular tangents is constant ;

hence if  $p_1, p_2, \pi_1, \pi_2$  denote the distances of the foci from the tangents  $\Sigma 1/p_1^2 = \text{constant}$ .

27. In Ex. 26, if the square of the radius of reciprocation is the power of the point with respect to the circle.

$p_1^2 + p_2^2 + \pi_1^2 + \pi_2^2 = \text{constant}$  ;  
or the locus of the intersection of rectangular tangents is a *concentric circle* (*Director Circle*).

28. From the properties of the conic, rectangular tangents, director circle, centre and line at infinity.

A variable chord of a conic which subtends a right angle at any point envelopes a conic ; and the focus and directrix of the envelope are pole and polar with respect to the given conic.

If the point is on the given conic *the envelope reduces to a point* \* on the perpendicular to the tangent passing through its point of contact. (*The Normal*.)

29. The base  $BC$  of a triangle  $ABC$  inscribed in a circle is fixed and the origin taken at its pole. Applying the formula of Art. 79, Ex. 10, we have the area of the reciprocal triangle constant, hence :—*the area cut off by any tangent with the asymptotes is constant.* And conversely, *given the vertical angle in position and area of a triangle, the envelope of the base is a conic ; and the sides are divided equianharmonically by the extremities of the base.*

30. Show by reciprocating from a vertex of a self conjugate triangle with respect to a circle that

$\alpha^\circ$ . The sum of the squares of any two conjugate diameters of an ellipse is constant.

$\beta^\circ$ . The difference of the squares of any two conjugate diameters of a hyperbola is constant.

31. Find by the methods of Art. 79, Exs. 3 and 4, the tangential equations of a conic circumscribed or inscribed to the triangle of reference.

\* This is proved independently as follows: If two right lines are drawn at right angles through a fixed point and intercept a variable segment  $AB$  on a fixed tangent to a circle ; the locus of the intersection of tangents through  $A$  and  $B$  is a line.

For it is a locus that can only meet the given tangent in one point ; therefore, etc., by reciprocation.

## CHAPTER VIII.

### SECTION I.

#### COAXAL CIRCLES.

82. **Definitions.**—The *Radical Axis*  $L$  of two circles  $A$ ,  $r_1$  and  $B$ ,  $r_2$  is the line perpendicular to  $AB$  and dividing it so that  $AL^2 \sim BL^2 = r_1^2 \sim r_2^2$ . Cf. Art. 72, Ex. 3.

It follows from the definition that  $L$  is the common chord of the circles when they intersect, and we may generalize this statement by regarding the radical axis as their chord of intersection real or imaginary.

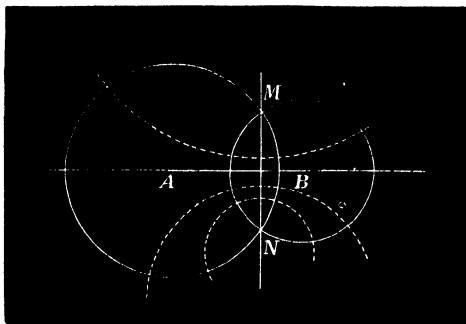
Thus all circles having a common radical axis pass through two real or two imaginary points.

Such a group is termed a *Coaxal System*.

83. It has been seen, Art. 72, Ex. 3, that a variable circle cutting two given ones orthogonally passes through two fixed points, viz., their common pair of inverse points; this orthogonal system is therefore coaxal; and from their mutual relations the two groups are said to be *Conjugate Coaxal Systems*. It is obvious that if either set possesses real points of intersection, the other does not; also the common points of one set are the common pair of inverse points with respect to the other Art. 72, Ex. 1.

Since the line of centres  $AB$  bisects the common chord

$MN$  it is the axis of reflexion of each common point with respect to the other.

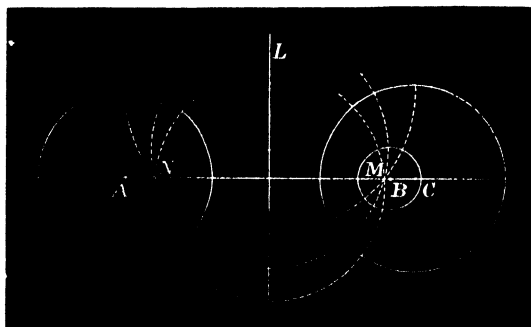


NOTE.—If two circles are concentric their radical axis is the line at infinity ; therefore a system of concentric circles passes through two imaginary points at infinity.

These are called the *Circular Points*.

If the circles touch, their radical axis is the common tangent at the point of contact.

If the circles reduce to points, the radical axis of two points is their axis of reflexion.



84. Let  $A, r_1; B, r_2; C, r_3 \dots$  denote the circles of a coaxal system. Then, since

$$AL^2 - BL^2 = r_1^2 - r_2^2, \quad AL^2 - CL^2 = r_1^2 - r_3^2, \text{ etc.,}$$

we have by transposing

$$AL^2 - r_1^2 = BL^2 - r_2^2 = CL^2 - r_3^2 = \dots = \pm k^2 \dots \dots (1)$$

The common value of these quantities ( $\pm k^2$ ) is the *Modulus* of the system. It is *positive* for a non-intersecting system and *negative* for the intersecting or common point species.

85. It follows from Art. 84 (1) that the position of the centre *C* of any circle of given radius of a coaxal system is determined, and conversely. In the former case

$$CL^2 = AL^2 - r_1^2 + r_3^2 = \text{a known quantity.}$$

Two values of *CL* equal in magnitude but of opposite signs are thus found. Hence the reflexion of every circle of the system with respect to the radical axis is also a circle of the system. The radical axis is therefore the line around which the entire group is symmetrically disposed.

86. The radical axes of three circles taken in pairs are concurrent (Art. 72, Ex. 6). In the particular case when their centres are collinear the axes are parallel, and the point of concurrence (*Radical Centre*) is at infinity. If the circles are coaxal the radical axes coincide and the tangents from any point on this line to the three circles are therefore equal.

Conversely, if three circles whose centres are collinear have a radical centre not at infinity they form a coaxal system.

87. *Limiting Values of the Radius* given by the equation  $AL^2 - r_1^2 = \text{const.}$

Since  $AL^2 - r_1^2$  is constant, *AL* and  $r_1$  increase and diminish in value together; or according as the centre

approaches to or recedes from the radical axis, the radius diminishes or increases.

It follows in the limit when  $C$  is at infinity that the circle loses its curvature, and a portion of it coincides with the radical axis. The remainder being at infinity is the line at infinity; hence *we regard the line at infinity, and the radical axis, together as forming the circle of the system whose radius is infinitely great.\**

$$\begin{aligned} \text{Again, since } AL^2 - r_1^2 &= CL^2 - r_3^2 \text{ if } r_3 = 0, \\ CL^2 &= AL^2 - r_1^2 \dots \dots \dots (1) \end{aligned}$$

The two values of  $CL$  in this equation determine therefore the positions of the centres of the circles of infinitely small radii. These are the *Points* or *evanescent Circles* of the group, and are termed the *Limiting Points*.

$$\begin{aligned} \text{By (1) } r_1^2 &= AL^2 - CL^2 = (AL - CL)(AL + CL) \\ &= AC \cdot AC', \end{aligned}$$

where  $C'$  is the reflexion of  $C$  with respect to the radical axis; therefore the limiting points are the common pair of inverse points of the coaxal system. (Cf. Art. 72, Ex. 1.) Hence the radical axis of a circle and point is the axis of reflexion of the point and its inverse with respect to the circle.

**88. Theorems.—I.** *The radical axis of a coaxal system is the locus of a point the tangents from which to the circles are equal.*

Let the tangents from  $P$  be  $t_1$  and  $t_2$ .

\* Since two circles meet on their radical axis, we infer that any two circles pass through two imaginary points on the line at infinity. Also, because every two circles intersect on this line, therefore *all circles pass through the same two imaginary points, i.e. the Circular Points at Infinity.*

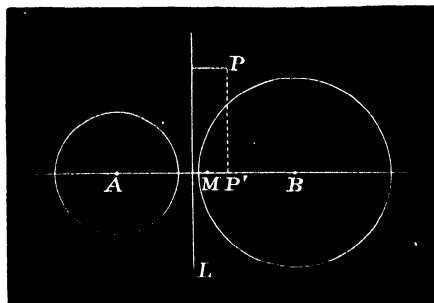
Then  $t_1^2 = PA^2 - r_1^2$ ,  $t_2^2 = PB^2 - r_2^2$ ;  
 hence, by subtraction,  
 $t_1^2 - t_2^2 = PA^2 - PB^2 - (r_1^2 - r_2^2) = 0$ ; (Art. 82)  
 therefore, etc.

**II.** More generally, *The difference of the squares of the tangents ( $t_1^2 \sim t_2^2$ ) from any point  $P$  to two circles = twice the rectangle under the distance between their centres and the distance of  $P$  from their radical axis; or*

$$t_1^2 - t_2^2 = 2AB \cdot PL.$$

For, draw  $PP'$  perpendicular to  $AB$  and take  $M$  the middle point of  $AB$ .

Then  $t_1^2 = AP^2 - r_1^2$ , and  $t_2^2 = BP^2 - r_2^2$ ;  
 hence  $t_1^2 - t_2^2 = AP^2 - BP^2 - (r_1^2 - r_2^2)$   
 $= AP^2 - BP^2 - (AL^2 - BL^2)$  (Euc. I. 47)  
 $= 2AB \cdot P'M + 2AB \cdot ML$ ; (Euc. II. 5 or 6)  
 therefore  $t_1^2 - t_2^2 = 2AB \cdot PL$ .



**COR. 1.** If  $P$  be any point on one of the circles ( $B, r_2$ ),

$$t_2 = 0, \text{ and } t_1^2 = 2AB \cdot PL, \text{ or } t_1^2 \propto PL;$$

or, *if the square of the tangent from a variable point to a given circle varies as its distance from a fixed line,*



*the locus of the point is a circle coaxal with the given circle and line.*

COR. 2. More generally, if  $C$  be the centre of a circle coaxal with  $A$  and  $B$  passing through  $P$ ,  $t_1$  and  $t_2$  the tangents from  $P$ , we have, by Cor. 1,

$$t_1^2 = 2AC \cdot PL \quad (1) \quad \text{and} \quad t_2^2 = 2BC \cdot PL \quad (2);$$

dividing (1) by (2), we have

$$\frac{t_1^2}{t_2^2} = \frac{AC}{BC}; \dots\dots\dots (3)$$

*hence the locus of point such that the ratio of the tangents drawn from it to two circles is constant is a coaxal circle whose centre is determined by (3).*

COR. 3. *The common tangents to two circles each subtend right angles at the limiting points.*

For, if  $M$  be a limiting point,  $XY$  one of the common tangents, and  $L$  its intersection with the radical axis,  $LX = LY = LM$ ; therefore, etc.

COR. 4. If a variable chord  $XY$  of a circle be divided at  $P$  such that  $PX \cdot PY \propto PM^2$ , where  $M$  is a fixed point; the locus of  $P$  is a circle coaxal with the given circle and point.

The line  $PM$  is the tangent from  $P$  to the limiting point  $M$ ; therefore, etc.

#### EXAMPLES.

1. If a variable chord ( $AB$ ) of a circle ( $O$ ,  $r$ ) subtend a right angle at a fixed point ( $M$ ), the loci—

$\alpha^\circ$ . of its middle point  $N$ ;

$\beta^\circ$ . of  $N'$  the foot of the perpendicular on it from  $M$ ;

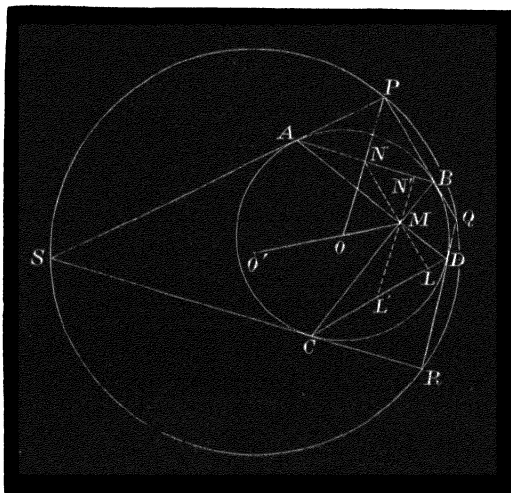
$\gamma^\circ$ . of the pole  $P$  of  $AB$

are circles each coaxal with the given point and circle.

[To prove  $\alpha^\circ$  and  $\beta^\circ$ ; we have

$$\frac{NM^2}{NA \cdot NB} = \frac{N'M^2}{N'A \cdot N'B} = -1;$$

hence  $N$  and  $N'$  lie on the same circle coaxial with  $M$  and  $O$ ,  $r$ , whose centre bisects internally the interval  $OM$ , by Cors. 2 and 4.



To prove  $\gamma^\circ$ . Since  $N$  describes a circle, its inverse  $P$  describes a circle coaxial with  $O$ ,  $r$  and the locus of  $N$ . For the locus of  $P$  is a circle; and it is coaxial with the other two, because the three circles have a common pair of points real or imaginary.]

2. The orthocentre of a triangle is the radical centre of the circles described on the sides as diameters; and the common value (Art. 77) of the rectangles under the segments of the perpendiculars is the radical product of the point with respect to the circles.

3. The middle points of the four common tangents to two circles the collinear.

[Each point of bisection is on the radical axis.]

4. Find the radical centre and product of the ex-circles of a given triangle.

[The middle point of the base is the middle point of the common tangent to the two circles which touch the base externally; therefore the line through it parallel to the internal bisector of the vertical angle, *i.e.* at right angles to their line of centres, is their radical axis. Similarly for each of the remaining pairs. Hence the radical centre is the in-centre of the median triangle; and, generally, the ex-centres of the median triangle are the radical centres of the three triads of circles formed by taking the in-circle and two ex-circles of the original triangle.]

For the values of the radical products, see Art. 48, Ex. 1.]

5. The circum-centre of a triangle is the radical centre of any three coaxal systems which have  $B$  and  $C$ ,  $C$  and  $A$ ,  $A$  and  $B$  for limiting points.

6. The extremities of any two secants to two given circles which intersect on their radical axis are concyclic.

7. Any circle  $P, R$  cutting two circles  $A, r_1$ ;  $B, r_2$  at angles  $\alpha$  and  $\beta$  meets the radical axis at an angle  $\theta$  given by the equation

$$\cos \theta = \frac{r_1 \cos \alpha - r_2 \cos \beta}{AB}.$$

[Denote the secants by  $PXX'$  and  $PYY'$ . Applying the formula  $t_1^2 - t_2^2 = 2AB \cdot PL$ , we have

$$\begin{aligned} 2AB \cdot PL &= R(R + XX') - R(R + YY') \\ &= R(XX' - YY') = 2R(r_1 \cos \alpha - r_2 \cos \beta); \end{aligned}$$

hence 
$$\frac{PL}{R} = \frac{r_1 \cos \alpha - r_2 \cos \beta}{AB}.$$

But  $PL/R$  = the cosine of the angle in the segment of  $P, R$  made by the intercept on the radical axis; therefore, etc.]

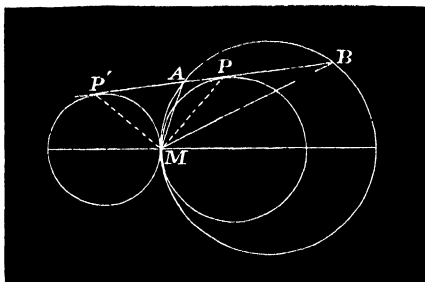
8. The axis of perspective of  $ABC$  and its pedal triangle is the radical axis of the circum- and nine-points-circles.

[By Art. 88, I. and Euc. III. 36.]

8a. The line joining the orthocentre and circum-centre is at right angles to the axis of perspective of  $ABC$  and the pedal triangle.

[It is the line of centres of the circum- and nine-points-circles.]

9. Two circles touch at  $M$  and a chord  $AB$  of either touches the other at  $P$ ; prove that  $PM$  is a bisector of the angle  $AMB$ .



[By Art. 88, Cor. 2,  $AP/AM = BP/BM$ .]

10. For any cyclic quadrilateral whose diagonals intersect in  $M$ ; prove that, if the bisectors of the angles between the diagonals meet the four sides in  $X, Y, X', Y'$ ,

$$AL \cdot BL \cdot CL \cdot DL = XL \cdot YL \cdot X'L \cdot Y'L,$$

where  $L$  is the radical axis of the circle and point.

11. If  $L, M, N$  denote the radical axes of three pairs of circles  $X$  and  $A$ ,  $X$  and  $B$ ,  $X$  and  $C$ , and  $L', M', N'$  the radical axes of  $Y$  and  $A$ ,  $Y$  and  $B$ ,  $Y$  and  $C$ ; to prove that the two triangles  $LMN$  and  $L'M'N'$  are in perspective; and that the centre of perspective is the radical centre of  $A, B$  and  $C$ ; and their axis of perspective the radical axis of  $X$  and  $Y$ .

[For  $MN$  is a point on the radical axis of  $B$  and  $C$  (Art. 72, Ex. 6); similarly  $M'N'$  a vertex of the triangle  $L'M'N'$  is on the same line; therefore, etc.]

12. If three lines  $AX, BY, CZ$  be drawn from the vertices of a triangle to the opposite sides; the radical centre of the circles on these lines as diameters is the orthocentre and their common orthogonal circle the polar circle of the triangle.

[The perpendiculars of the triangle are respectively chords of these circles; therefore, etc. Art. 77.]

13. For any three circles  $A, B, C$  and three others taken with them such that  $B, C, X$ ;  $C, A, Y$ ;  $A, B, Z$  form three coaxal systems; to prove that,  $1^\circ$ , the system of six circles have the same radical

centre and product; and, 2°, if the centres of  $X, Y, Z$  are collinear, these circles are coaxal.

[In 1° the radical centre and product is obviously that of the circles  $A, B, C$ ; 2° follows at once since, if the circles be not coaxal, their radical centre is at infinity. Art. 86.]

14. Two coaxal systems have a common circle; find the locus of the points of contact of the circles which touch.

[Let  $L$  and  $L'$ , the radical axes of the systems, meet at  $P$ , and  $T$  be one point of contact. The common tangent at  $T$  passes through  $P$ , and  $PT$  is the radius of the common orthogonal circle of the two systems, which is therefore the required locus.]

15. The radical axis of any two circles bisects the distance between the polars of the centre of each circle with respect to the other.

\*16. Three circles are described each touching two sides of a triangle and the circum-circle internally in points  $L, M$ , and  $N$ ; to prove that the triangles  $ABC$  and  $LMN$  are in perspective.

[Let one of the circles touch the sides  $a$  and  $b$  in the points  $P$  and  $Q$  and the circum-circle in  $N$ . Then  $N$  being a limiting point of the two circles  $AQ^2/AN^2 = BP^2/BN^2 = (R - \rho)/R$ , where  $\rho$  is the radius of the inner circle; but  $AQ = b - CQ = b - ab/s$ , Art. 6, Ex. 3; similarly,  $BP = a - ab/s$ ; substituting these values and reducing we get  $\frac{AN}{BN} = \frac{s-a}{a} / \frac{s-b}{b}$ . Also,  $AN/BN$  = the ratio of the perpendiculars from  $N$  on the sides  $b$  and  $a$  respectively. (Euc. III. 22.)

Similarly, the ratios of the perpendiculars from  $L$  and  $M$  on the corresponding pairs of sides of  $ABC$  are  $\frac{s-b}{b} / \frac{s-c}{c}$  and  $\frac{s-c}{c} / \frac{s-a}{a}$ ; therefore, etc., by Art. 65.]

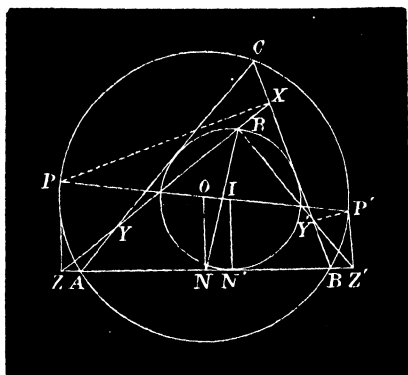
\*17. If circles are described as in Ex. 16 touching the circum-circle externally in points  $L', M', N'$ , the triangles  $ABC$  and  $L'M'N'$  are in perspective.

\*18. The centres of perspective in Exs. 16 and 17 are respectively the isogonal conjugates of the centres of perspective of  $ABC$  and

the triangle formed by joining the internal points of contact of the escribed circles with the sides (*point de Nagel*) ; and of  $ABC$  and the triangle formed by joining the points of contact of the in-circle with the sides (*point de Gorgonne*).

[Make use of the property given in Art. 64, Ex. 3.]

19. The nine-points-circle of a triangle touches the in- and three ex-circles.



[Let  $ABC$  be the triangle,  $O$  and  $I$  the centres of the circum- and in-circles,  $PP'$  the common diameter,  $XYZ$  and  $X'Y'Z'$  the Simson lines of  $P$  and  $P'$ ,  $R$  their point of intersection,  $L, M, N$  the middle points of the sides,  $L', M', N'$  the points of contact of the in-circle.

Since  $OP = OP'$ ,  $NZ = NZ'$ . But the Simson lines of two points diametrically opposite meet at  $R$  at right angles on the nine points circle ; therefore  $NZ = NZ' = NR$ . Again,  $OP/OI = NZ/NN' = NR/NN'$  ; therefore  $NR/NN' = MR/MM' = LR/LL'$  ; hence it follows that  $R$  is a limiting point of the in-circle and the circum-circle of the triangle  $LMN$ . See Art. 83 Note. This elegant proof of the well-known property is due to M'Cay.]

20. A variable circle  $O, \rho$  touches two circles  $A, r_1$  ;  $B, r_2$  ; prove that the polar  $M$  of its centre with respect to either  $(A, r)$  envelopes a fixed circle.

[Since it touches the two circles, it cuts their radical axis  $L$  at a constant angle (Art. 88, Ex. 7), or  $\rho/OL = \text{const.}$  Draw a parallel







hence the circle coaxal with two given ones whose centre divides the distance between their centres in the ratio of the radii is the locus of a point, the tangents from which to the given circles are in the sub-duplicate ratio of the radii.

COR. 4. If the intercepts are equal,  $XX' = YY'$ , the tangents are in the ratio of the radii and the locus of their intersection is called the *Circle of Similitude* of the given ones; its centre  $C$  is given by the equation

$$AC/BC = r_1^2/r_2^2, \dots\dots\dots (\text{Cor. 1.}) (1)$$

COR. 5. Since  $AB$  is divided internally and externally in  $C_2$  and  $C_1$  such that  $\frac{AC_1}{BC_1} = \frac{AC_2}{BC_2} = \frac{r_1}{r_2}$  and again in  $C$ , by Cor. 4, such that  $\frac{AC}{BC} = \frac{r_1^2}{r_2^2}$ , it follows (Art. 70) that  $C$  is the middle point of the segment  $C_1C_2$  and that the circle of similitude is the circle on it as diameter.

COR. 6. If the line  $XX'YY'$  passes through the intersections ( $QS, PR$  and  $PS, QR$ ) of opposite connectors of the quadrilateral; when  $PQ$  and  $RS$  are parallel; the circles  $A$  and  $B$  reduce to points and are therefore the limiting points of the system; *i.e. the common pair of inverse points of the circum-circle of the trapezium PQRS and that touching the parallel sides at Z and Z'.* (Art. 72, Ex. 13.)

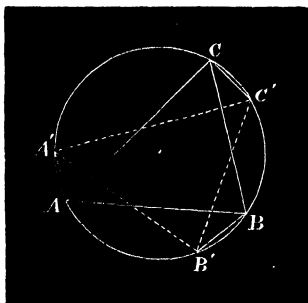
#### EXAMPLES.

1. Any line meeting a pair of opposite sides of a cyclic quadrilateral at equal angles makes equal angles with each of the remaining pairs (Euc. III. 21, 22); intersects them in points  $XX', YY', ZZ'$  such that the circles touching the pairs of opposite con-

nectors at these points are coaxal with the given one ; and one of them lies on the side of the radical axis opposite to the other two.

2. A variable quadrilateral inscribed in a circle moves so that a pair of opposites envelope a circle, then each of the remaining pairs of opposites always touch circles coaxal with the given ones.

3. A variable triangle  $ABC$  is inscribed in a circle of a coaxal system, and two of its sides each envelope a circle of the system ; to prove that the third side  $AC$  envelopes another.



[Let  $A'B'C'$  be any other position of the given triangle. Then  $ABA'B'$  is a cyclic quadrilateral, and one pair of opposites  $AB$  and  $A'B'$  touch a given circle, therefore  $AA'$  and  $BB'$  touch one circle of the system.

Similarly  $BB'$  and  $CC'$  touch one circle of the system. But  $BB'$  can touch only one circle of the group on either side of the radical axis, Art. 92, Ex. 6 ; hence  $AA'$ ,  $BB'$ ,  $CC'$  touch the same circle. Now consider the quadrilateral  $AA'CC'$  ; it is obvious by Ex. 2 that  $AC$  and  $A'C'$  touch one circle ; therefore the envelope of  $AC$  is a coaxal circle.\*]

4. **Poncelet's Theorem.**—If a variable polygon inscribed in a circle of a coaxal system moves so that all the sides but one touch fixed circles of the system, the last side also touches in every position a fixed circle of the system.

[By Ex. 3.]

---

\* Dr. Hart, *Quarterly Journal*, Vol. II. p. 143.

5. The problem "to describe a polygon having all its vertices on a given circle and all its sides touching another" is either impossible or indeterminate.

[Let all the circles in Ex. 4 touching all the sides but one of the polygon coincide; it follows therefore that if the last side touches this circle in one position it touches it in every position.]

6. To find the relation connecting the radii  $r_1$  and  $r_2$  of two circles and the distance  $\delta$  between their centres so that a quadrilateral may be inscribed to one and circumscribed to the other. (Art. 88, Ex. 1.)

[By Ex. 5, when this is possible the position of the quadrilateral is indeterminate. Assuming it to have the position of symmetry, i.e., with a pair of opposite vertices at the extremities of the common diameter, and  $\theta$  the angle between any side and this diameter. By right-angled triangles we have the relations

$$\frac{r_1}{r_2 - \delta} = \sin \theta \text{ and } \frac{r_1}{r_2 + \delta} = \cos \theta$$

squaring and adding these results

$$\frac{1}{(r_2 - \delta)^2} + \frac{1}{(r_2 + \delta)^2} = \frac{1}{r_1^2}.]$$

7. If  $A, r_1, B, r_2, C, r_3$  be three coaxal circles such that a variable quadrilateral whose pairs of opposite sides envelope  $A$  and  $B$  is inscribed in  $C$ , prove that

$$\frac{r_2^2}{(r_3 - \delta_1)^2} + \frac{r_1^2}{(r_3 + \delta_2)^2} = 1$$

where  $\delta_1$  and  $\delta_2$  denote the distances  $AC$  and  $BC$ .

[By the method of Ex. 6.]

8. If a variable line  $L$  meet two circles  $Ar_1, Br_2$  so that the chords intercepted,  $2c$  and  $2c'$  are in a constant ratio  $\kappa$ ; to show that two points  $A', B'$  may be found on the line  $AB$  to satisfy the relation

$$A'L \cdot B'L = \text{const.}$$

[For

$$c^2 = r_1^2 - AL^2, \quad c'^2 = r_2^2 - BL^2,$$

hence

$$r_1^2 - AL^2 = \kappa^2(r_2^2 - BL^2),$$

or

$$(AL + \kappa BL)(AL - \kappa BL) = \text{const.},$$

but

$$AL + \kappa BL = (1 + \kappa)A'L,$$

and

$$AL - \kappa BL = (1 - \kappa)B'L,$$

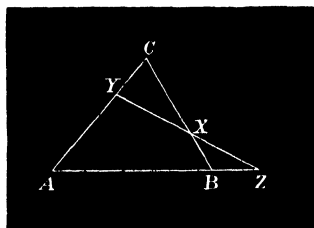
where  $A'$  and  $B'$  divide the line  $AB$  internally and externally in the ratio  $\kappa : 1$ .]

NOTE.—The variable line in the present article is thus seen to envelope a conic of which the points  $A'$  and  $B'$  are the foci.

90. We have seen, Art. 86, that in general three circles have but one common orthogonal circle, and in the particular case when more than one can be drawn the three form a coaxal system.

This property is sometimes of use in determining whether circles are coaxal, and may be regarded as a criterion of coaxality. The following illustrations are due to Walker.

91. Let  $ABC$  be a triangle and  $XYZ$  any transversal to its sides. Join  $AX$ ,  $BY$ ,  $CZ$ . These lines are drawn from the vertices of each of the four triangles  $AYZ$ ,  $BZX$ ,  $CXY$ ,  $ABC$ , and terminated by the opposite sides; therefore, Art. 88, Ex. 12, the orthocentres of the four triangles are each the radical centre of the circles described on  $AX$ ,  $BY$ ,  $CZ$  as diameters.



Hence we have the following theorems:—

1°. The orthocentres of the four triangles formed by any four lines are collinear.

2°. The middle points of the diagonals  $AX$ ,  $BY$ ,  $CZ$  of a complete quadrilateral are collinear.

3°. The line of collinearity of the orthocentres is at right angles to the line in 2°, called the *Diagonal Line of the Quadrilateral*.

4°. The circles on the three diagonals as diameters are coaxal.

5°. The polar circles of the four triangles belong to the conjugate coaxal system.

#### EXAMPLES.

1.  $A$ ,  $B$ ,  $C$ ,  $D$  are the vertices of a convex quadrilateral taken in order;  $A_e$ ,  $B_e$ ,  $C_e$ ,  $D_e$  and  $A_i$ ,  $B_i$ ,  $C_i$ ,  $D_i$  the external and internal bisectors of the angles; prove that

$\alpha^\circ$ . The sixteen centres of the circles touching the sides of the four triangles formed by taking the sides of the quadrilateral in triads, lie in fours on these bisectors.

$\beta^\circ$ . The following groups of quadrilaterals are cyclic:—

$$\begin{array}{ll} A_e B_i C_i D_e \} (a) & A_i B_i C_e D_e \} (c) \\ A_i B_e C_e D_i \} & A_e B_e C_i D_i \} \\ A_i B_e C_i D_e \} (b) & A_e B_e C_e D_e \} (d) \\ A_e B_i C_e D_i \} & A_i B_i C_i D_i \} \end{array}$$

$\gamma^\circ$ . Groups (a) and (c) are coaxal, and groups (b) and (d) conjugately coaxal.

[These properties are proved by employing Euc. III. 32 to show that any circle of either group is cut orthogonally by any circle of the other group. Russell.]

2\*.  $A$ ,  $B$ ,  $C$ ,  $D$  are four points on a circle. Omitting each point in turn we have four triangles; prove that the sixteen centres of the circles touching the sides of these triangles lie in fours on four parallel lines, and also in fours on four lines each perpendicular

to the former set; and that the two sets of lines are parallel to the bisectors of the angle between  $AC$  and  $BD$ . (M'Cay.)

3.  $ABC$  is a triangle,  $AA'$  a diameter of the circum-circle and  $H$  the orthocentre; show that  $A'$  and  $H$  are equidistant from the base  $BC$ ; and hence deduce the theorem "the Simson line of any point is equidistant from the point and orthocentre of the triangle."

## SECTION II.

### ADDITIONAL CRITERIA OF COAXAL CIRCLES.

92. I. *Relation connecting the distances between the centres and the radii of three circles of a coaxal system.*

Let the circles be denoted by  $A, r_1; B, r_2; C, r_3$ .

Then for any point  $P$  on the radical axis, we have

$$BC \cdot AP^2 + CA \cdot BP^2 + AB \cdot CP^2 = -BC \cdot CA \cdot AB;$$

hence if  $t$  be the length of the tangent from  $P$  to the circles, since  $AP^2 = r_1^2 + t^2$ , etc., by substituting in this equation and reducing,

$$BC \cdot r_1^2 + CA \cdot r_2^2 + AB \cdot r_3^2 = -BC \cdot CA \cdot AB, \dots (1)$$

a result from which the radius  $r_3$  of any circle of the system may be found when the position of its centre is known; and conversely.

COR. 1. If  $r_3 = 0$ ,  $C$  is a limiting point (Art. 87), by letting  $AC = x$  in (1) we obtain a quadratic in  $x$ , the last term of which is  $r_1^2$ . Hence the *product of the distances of the limiting points from the centre of any circle of the system = the square of its radius*. Cf. Art. 87.

COR. 2. If  $r_2 = r_3 = 0$ , the criterion reduces to

$$AB \cdot AC = r_1^2.$$

## EXAMPLES.

1. If  $t_1, t_2, t_3$  denote the tangents from any point  $P$  to three circles of a coaxial system ; to prove that

$$BC \cdot t_1^2 + CA \cdot t_2^2 + AB \cdot t_3^2 = 0.$$

[For  $BC \cdot AP^2 + CA \cdot BP^2 + AB \cdot CP^2 = -BC \cdot CA \cdot AB$ ,.....(1)

and  $BC \cdot r_1^2 + CA \cdot r_2^2 + AB \cdot r_3^2 = -BC \cdot CA \cdot AB$ .....(2)

Subtracting (2) from (1) ; therefore, etc.]

2. Deduce as a particular case of Ex. 1 the theorem :—The locus of a point, the tangents from which to two given circles are in a constant ratio, is a coaxial circle.

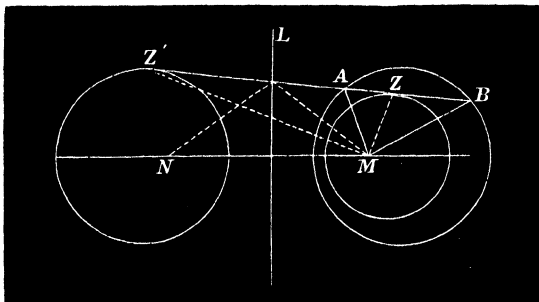
[Let  $t_3=0$ .]

3. Explain the formula of Ex. 1 when  $t_2=t_3=0$ .

4. Find the locus of a point  $P$  if the product of the tangents from it to two circles bears a constant ratio to the square of the tangent to any circle coaxial with them ( $kt_1t_2=t_3^2$ ).

[In Ex. 1, substituting the given condition, the equation reduces to the form  $(t_1 - mt_2)(t_1 - nt_2) = 0$  ; hence  $P$  describes two coaxial circles, since the ratio of the tangents  $t_1$  and  $t_2 = m$ , or  $n$ .]

5. If the common tangent  $ZZ'$  to two circles meet a coaxial circle in the points  $A$  and  $B$  ; to prove that  $MZ$  and  $MZ'$  are the bisectors of the angles subtended by the chord  $AB$  at either limiting point.



[For  $AZ, AM$  and  $BZ, BM$  being pairs of tangents drawn from two points  $A$  and  $B$  on the same circle to two circles of the system, it follows that  $AZ/AM = BZ/BM$ , by alternation  $AM/BM = AZ/BZ$ , and for a similar reason  $= AZ'/BZ'$  ; therefore, etc.]

6. To describe two circles of a coaxal system touching a given line.

[In Ex. 5 divide the line  $AB$  internally and externally in  $Z$  and  $Z'$  in the given ratio  $AM/BM$ ; therefore  $Z$  and  $Z'$  are the required points of contact. It will be noticed that the circles lie one on each side of the radical axis.]

7. A triangle  $ABC$  is inscribed in a circle of a coaxal system; prove that the points of contact  $X, X', Y, Y', Z, Z'$  of the three pairs of circles of the system which touch the sides  $BC, CA$ , and  $AB$  respectively,

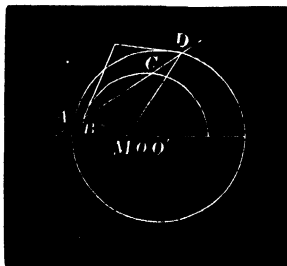
$\alpha^\circ$ . Lie three and three on four lines,

$\beta^\circ$ . Connect with the opposite vertices by six lines, passing three and three through four points.

[Apply the relations in Ex. 5 to the three sides; therefore, etc. Arts. 62 and 63.]

8. Apply the criterion of the Article to show that the nine-points-, circum- and polar circles are coaxal.

9. If points  $B$  and  $D$  are taken on any two circles whose centres are  $O$  and  $O'$  and joined to the limiting point  $M$  such that  $BMD$  is a right angle, the locus of the intersection of tangents at  $B$  and  $D$  to the circles is a coaxal circle.



[Let the line  $BD$  meet the circles again in  $A$  and  $C$ ; then

$$\frac{MB^2}{AB \cdot BD} = \frac{MO}{OO'} = \frac{MC^2}{AC \cdot CD} = \frac{MB \cdot MC}{(AB \cdot AC \cdot BD \cdot CD)^{\frac{1}{2}}};$$

also, 
$$\frac{MA^2}{AB \cdot AC} = \frac{MO'}{OO'} = \frac{MD^2}{BD \cdot CD} = \frac{MA \cdot MD}{(AB \cdot AC \cdot BD \cdot CD)^{\frac{1}{2}}}.$$



Whence, 
$$\frac{MB \cdot MC}{MA \cdot MD} = \frac{MO}{MO'} \dots \dots \dots (1)$$

But since  $BMD = 90^\circ$ ,  $AMC = 90^\circ$  (Art. 72, Cor. 8),  
and therefore  $BMC + AMD = 180^\circ$ ;

hence 
$$\frac{BC}{AD} = \frac{MB \cdot MC}{MA \cdot MD} = \frac{MO}{MO'}$$

by (1), a constant quantity ; therefore, etc. (cf. Art. 89, Ex. 8).]

10. A quadrilateral  $PQRS$  is inscribed to one circle and escribed to another at the points  $A, B, C, D$  ; prove that its position is *indeterminate*, and the diagonals  $PR$  and  $QS$ ,  $BC$  and  $AD$  of the two cyclic quadrilaterals intersect (the latter at right angles) at the limiting point  $M$ .

[By Art. 89, Ex. 6. See also Art. 88, Ex. 1, and Art. 67, Cor. 6.]

11. Construct a quadrilateral in a given circle symmetrical with respect to a given diameter and circumscribed to a circle having its centre at a fixed point on the diameter.

[Find the radius of the second circle by Art. 89, Ex. 6.]

93. II. *A variable circle cuts three others of a coaxal system at angles  $\alpha, \beta, \gamma$ , to prove the relation*

$$BC \cdot r_1 \cos \alpha + CA \cdot r_2 \cos \beta + AB \cdot r_3 \cos \gamma = 0.$$

Let  $P, \rho$  be the variable circle meeting the given ones at the points  $R, S, T$  respectively ; join  $PR, PS, PT$ , and produce the lines to meet the circles again in  $R', S', T'$ .

By Art. 92, Ex. 1,  $BC \cdot t_1^2 + CA \cdot t_2^2 + AB \cdot t_3^2 = 0$ , but  $t_1^2 = PR \cdot PR' = \rho(\rho + RR') = \rho(\rho + 2r_1 \cos \alpha)$ , with similar values for  $t_2$  and  $t_3$ . Substituting these values in the equation and reducing, we obtain the required result.

COR. 1. If two of the circles are cut orthogonally, every circle of the system is cut orthogonally. For if  $\alpha = \beta = 90^\circ$ , two terms of the equation vanish, therefore  $AB \cdot r_3 \cos \gamma = 0$  or  $\gamma = 90^\circ$ .

COR. 2. If the variable circle touch two of the given ones, it cuts the circle  $C, r_3$  coaxal with them at an angle

determined by the equation  $AB.r_3\cos\gamma = \pm BC.r \pm CA.r_2$ ; like signs being taken when the contacts are similar and unlike signs when the contacts are dissimilar. The four possible values arising from the selections of sign on the right side of the equation give the values of  $\gamma$  corresponding to each assigned species of contact.

COR. 3. In Cor. 2, if  $\cos\gamma = 0$ , the centres  $C$  of the particular circles of the system which are cut at right angles are given by the relation

$$BC.r_1 \pm CA.r_2 = 0,$$

or

$$AC/BC = \pm r_1/r_2.$$

Hence, the variable circle having similar contacts with two given circles cuts at right angles the coaxial circle whose centre is their external centre of similitude; and, if the contacts are dissimilar, the coaxial circle whose centre is the internal centre of similitude.

COR. 4. If  $\alpha = \pm\beta$  and  $\gamma = 90$ , the equation reduces to  $AC/BC = \pm r_1/r_2$ , as in Cor. 3. Hence, *the variable circle cutting two others at equal or supplemental angles cuts at right angles their external or internal circle of anti-similitude respectively.*

COR. 5. Let the radius of the variable circle be infinite; hence (Cor. 3) all lines cutting two circles at equal or supplemental angles are diameters of their external or internal circles of antisimilitude.

#### EXAMPLES.

1. To describe a circle cutting any three circles  $A, r_1$ ;  $B, r_2$ ;  $C, r_3$  at given angles  $\alpha, \beta, \gamma$ .

[The required circle cutting  $B, r_2$ ;  $C, r_3$  at given angles, therefore touches a known circle coaxial with them by Cor. 2; similarly for each of the remaining pairs of the given circles; hence the problem

reduces to “describe a circle touching three given circles with assigned contacts.” There are in consequence eight solutions. These are given in a subsequent chapter.]

2. Show that Ex. 1 cannot be reduced to describing a circle cutting three given circles orthogonally.

[For let  $X$  be the circle coaxal with  $B$  and  $C$  which is cut orthogonally by the required circle, and constructed by putting  $\gamma=90$  in the relation of the present Article; similarly let  $Y$  coaxal with  $C$  and  $A$ , and  $Z$  coaxal with  $A$  and  $B$ , be circles cut orthogonally by it. Their centres, being found by the relations

$$\frac{BX}{CX} = \frac{r_3 \cos \gamma}{r_2 \cos \beta}, \quad \frac{CY}{AY} = \frac{r_1 \cos \alpha}{r_3 \cos \gamma}, \quad \frac{AZ}{BZ} = \frac{r_2 \cos \beta}{r_1 \cos \alpha},$$

are collinear, Art. 62, and their common orthogonal circle therefore indeterminate.]

3. A variable circle  $P$ ,  $\rho$  touches two others  $A$ ,  $r_1$ ;  $B$ ,  $r_2$ ; show that the square of the common tangent  $t$ , to it and any third circle  $C$ ,  $r_3$  coaxal with them, varies as its radius ( $t^2 \propto \rho$ ).

[By Cor. 2 it cuts  $C$ ,  $r_3$  at a constant angle  $\gamma$ . But (Art. 4 (1))  $4 \sin^2 \frac{1}{2} \gamma = t^2 / \rho \cdot r_3$ ; therefore, etc. In the particular case when  $C$ ,  $r_3$  is a limiting point we have the theorem:—“if a variable circle touch two fixed circles, its radius is in a constant ratio to the square of the tangent to it from either of the limiting points.” Also, “the ratio of the tangents from the limiting points is constant.”]

4. A variable circle cuts two fixed circles at angles  $\alpha$  and  $\beta$ , tangents are drawn from its centre to the circles, and tangents  $t_1$  and  $t_2$  from the points of contact to the variable circle; prove that

$$t_1^2 / t_2^2 = r_1 \cos \alpha / r_2 \cos \beta,$$

and deduce the properties of Ex. 3 as particular cases (Preston). See *Spherical Trigonometry*, Art. 159, Ex. 15.

5. Find the locus of the centre of a circle cutting any three circles at equal or supplemental angles.

[By Cor. 4.]

6. The vertex and base of a triangle are fixed in position and the vertical angle given in magnitude; find the envelope of the circum-circle.

## SECTION III.

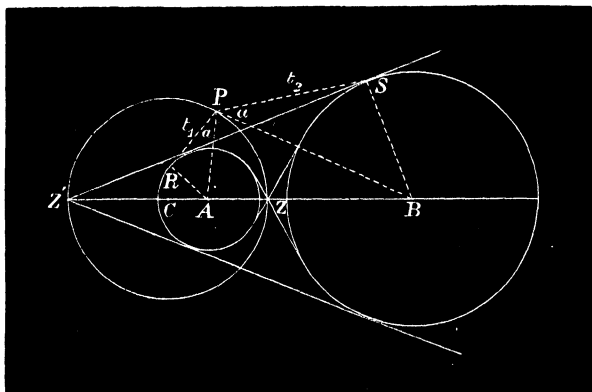
## CIRCLE OF SIMILITUDE.

94. Let  $A, r_1; B, r_2$  be any two circles,  $Z$  and  $Z'$  the points of section of  $AB$  such that

$$\frac{AZ}{BZ} = \frac{AZ'}{BZ'} = \frac{r_1}{r_2};$$

then the segments  $AB$  and  $ZZ'$  divide each other harmonically, and the circle  $C, r_3$  on  $ZZ'$  as diameter is termed their *Circle of Similitude*. The points  $Z$  and  $Z'$  are the *Internal* and *External Centres of Similitude*.

95. The circle of similitude has the following fundamental properties :—



1°. Its centre  $C$  and radius  $r_3$  are connected by the relation  $CA \cdot CB = r_3^2$  (Art. 70), or *the centres of the given circles are inverse points with respect to their circle of similitude*.

2°. The points  $Z$  and  $Z'$  are the intersections of the transverse and direct common tangents.

3°. It is coaxal with the given circles.

[For  $Z$  and  $Z'$  are on the same circle coaxal with  $A$  and  $B$ , since the ratios of the tangents from them are each equal to the ratio of the radii, and only one circle coaxal with  $A, r_1$  and  $B, r_2$  can contain these points, viz. that on the line  $ZZ'$  as diameter.]

4°. From Cor. 3 it is the locus of a point such that the tangents drawn from it to the circle have the constant ratio of the radii.

[Cf. Art. 88, Cor. 2.]

This follows independently, since  $PZ$  and  $PZ'$  are the bisectors of the angle  $APB$ , hence

$$PA/PB = AZ/BZ = AR/BS;$$

therefore, etc., by Euc. VI. 7.

5°. The circles subtend equal angles at any point on it. (By 4°.)

6°. In the particular case when the circle  $B, r_2$  becomes a right line the centre  $B$  is at infinity, its inverse  $A$  (Cor. 1) coincides with  $C$ , therefore *the centres of similitude of a line and circle are the extremities of the diameter of the circle perpendicular to the line.*

#### EXAMPLES.

1. The circles of similitude of any three circles taken in pairs are coaxal.

[Their centres are collinear, Art. 72, Ex. 21; therefore, etc., Art. 88, Ex. 13, 2°.]

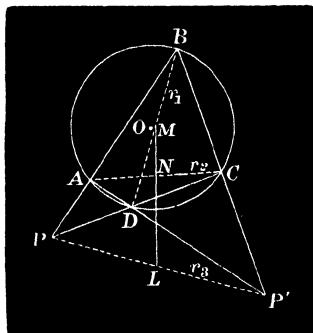
2. A circle cuts two at angles  $\alpha$  and  $\beta$ ; find the angle it makes with their circle of similitude.

3. The tangents from any point  $P$  on the circle of similitude to the circles  $A, r_1$  and  $B, r_2$  meet them at  $R$  and  $S$ ; prove ( $\alpha^\circ$ ) the chords which the circles intercept on the line  $RS$  are equal to one another; ( $\beta^\circ$ ) The tangents from  $R$  and  $S$  to the circles  $B$  and  $A$  are equal.

[Compare Art. 89, Cor. 4.]

4. The circle on the third diagonal of a complete cyclic quadrilateral is the circle of similitude of those described on the remaining two.

[Let  $ABCD$  be the quadrilateral,  $LMN$  its diagonal line,  $PP'$  the third diagonal,  $BD=2r_1$ ,  $CA=2r_2$ ,  $PP'=2r_3$ . Join  $PM$ ,  $PN$ .



The triangles  $PAC$  and  $PBD$  are similar, Euc. III. 21; hence, since  $PN$  and  $PM$  are homologous lines,  $PBM$  and  $PCN$  are similar; therefore  $PM/PN=r_1/r_2$ . Similarly,  $P'M/P'N=r_1/r_2$ ; therefore  $P$  and  $P'$  lie on a circle to which  $M$  and  $N$  are inverse points. Also the circles on the three diagonals are coaxial; therefore, etc. It follows also by  $1^\circ$  that  $LM \cdot LN=r_3^2$ .]

5. Having given the three diagonals of a cyclic quadrilateral; to construct it.

[Let  $O$  be the centre of the circle and  $r_1, r_2, r_3$  the diagonals. By Ex. 4  $LM \cdot LN=r_3^2$ , and is therefore known. Also  $LM/LN=r_1^2/r_2^2$ ; hence the lines  $LM$  and  $LN$  are determined.  $LM=r_1r_3/r_2$ ,  $LN=r_1r_2/r_3$ , and  $MN=\frac{r_3}{r_1r_2}\left(\frac{r_1^2-r_2^2}{r_1r_2}\right)$ . But  $OM$  and  $ON$  are known (Euc. I. 47), consequently the triangle  $OMN$  is completely determined.]

6. Six circles pass through two points  $P$  and  $Q$  on the circum-circle of a triangle  $ABC$  and touch the sides ; prove that the points of contact  $X, X'; Y, Y'; Z, Z'$  lie in threes on four lines.

[Let the line joining the points  $P$  and  $Q$  cut the sides of the triangle in  $L, M$ , and  $N$  respectively, and we have obviously  $LX = LX'$  and  $LB \cdot LC = LX^2 = LX'^2$ , with similar relations on the remaining sides of the triangle ; therefore, etc.]

7. From any point on a given line tangents are drawn to a circle ; a circle is described touching the fixed circle and variable pair of tangents to it ; prove that the envelope of the polar of its centre is a circle.

8. The circle of similitude of the circum- and nine-points-circle of a triangle is that described on the interval between the centroid and orthocentre as diameter.

[Let  $O$  be the circum-centre,  $H$  orthocentre,  $N$  the nine-points centre, and  $E$  the centroid. By a well-known property of these four collinear points  $OE/NE = OH/NH = 2 =$  ratio of radii of circum- and nine-points-circles ; therefore, etc.]

[It is called the *Orthocentroidal Circle* of the triangle.]

### MISCELLANEOUS EXAMPLES.

1. Prove that the equation of the two circles touching three given ones with contacts of similar species are

$$23\sqrt{S_1} + 31\sqrt{S_2} + 12\sqrt{S_3} = 0,$$

where  $S_1, S_2, S_3$  denote the powers of any point on either of the tangential circles with respect to the given ones.

2. If a variable chord  $AB$  of a circle is such that the sum of the tangents from  $A$  and  $B$  to another given circle is proportional to the length of  $AB$ , it envelopes a circle coaxal with the two.

3. If a variable circle touches two fixed circles and cuts a circle concentric with either in the points  $A$  and  $B$  : required to find the envelope of  $AB$ . (Dublin Univ. Exam. Papers, 1891.)

[Applying Casey's relation between the common tangents to four

circles to the points  $A$  and  $B$  and the two given circles, it follows by Ex. 2 that the envelope of  $AB$  is a coaxal circle.]

4. Prove that the circles cutting three given ones orthogonally passing through their circles, and bisecting the circumferences are coaxal.

5. Reciprocate the following theorem from a limiting point :—The square of the distance of any point on a circle from a limiting point varies as its distance from the radical axis.

[The rectangle under the distances of the foci from any tangent to a conic is constant.]

6. Prove that the limiting points of any two circles lie on a pair of opposite connections of their common escribed quadrilateral.

7. If  $\delta$  denote the distance between the limiting points and  $\gamma$  the length of their imaginary common chord, prove that  $\delta = i\gamma$ .

8. If two circles whose radii are  $r_1$  and  $r_2$  are so related that a hexagon can be inscribed to one and circumscribed to the other, then

$$\frac{1}{(r_1^2 - \delta^2)^2 + 4r_1r_2^2\delta} + \frac{1}{(r_1^2 - \delta^2)^2 - 4r_1r_2^2\delta} = \frac{1}{2r_2^2(r_1^2 + \delta^2) - (r_1^2 - \delta^2)^2}.$$

9. If an octagon can be inscribed to one and circumscribed to the other,

$$\left\{ \frac{1}{(r_1^2 - \delta^2)^2 + 4r_2^2r\delta} \right\}^2 + \left\{ \frac{1}{(r_1^2 - \delta^2)^2 - 4r_2^2r\delta} \right\}^2 \\ = \left\{ \frac{1}{2r_2^2(r_1^2 + \delta^2) - (r_1^2 - \delta^2)^2} \right\}^2.$$

10. The mean centre of the vertices of a cyclic quadrilateral lies on the circumference of the nine-points-circle of the harmonic triangle of the quadrilateral. (Russell.)

11. If a variable polygon is inscribed to one circle and escribed to another, the locus of the mean centre of any number ( $r$ ) of consecutive points of contact is a circle. (Weill). Cf. Art. 53, Ex. 12.

12. Prove the following extension of Weill's theorems :—If a variable polygon of any order be inscribed in a circle of a coaxal



system having all its sides touching respectively fixed circles of the system ; there exists a set of multiples for which the mean centre of the points of contact of the sides with the circles is a fixed point.

[Let any circle of the system be denoted by  $(O, r, \delta)$  where  $\delta$  is the distance of its centre from the circumcentre of the polygon, and let  $\alpha, \beta, \gamma$ , and  $c$  be the displacements of the points of contact of the sides  $AB, BC, CD$ , etc. for consecutive positions. Then, by Art. 53, Ex. 12, we have

$$\frac{\frac{\sqrt{\delta_1} \alpha}{r_1}}{AB} = \frac{\frac{\sqrt{\delta_2} \beta}{r_2}}{BC} = \frac{\frac{\sqrt{\delta_3} \gamma}{r_3}}{CD} = \text{etc.}$$

hence the mean centre of the points of contact remains fixed for the system of multiples  $\sqrt{\delta_1}/r_1, \sqrt{\delta_2}/r_2, \sqrt{\delta_3}/r_3$ , etc.]

12a. The locus of the mean centre of  $r$  consecutive points of contact for their respective multiples is a circle.

[For, join the extremities of the  $r$  sides thus forming a polygon of  $r+1$  sides, and let the last side touch a fixed circle  $(O_{r+1}, r_{r+1}, \delta_{r+1})$  of the system. (Art. 89, Ex. 4.) By Ex. 12, the mean centre of the  $r+1$  points of contact for the corresponding multiples is a fixed point ( $X$ ). Let  $Y$  be the mean centre for the  $r$  points and  $Z$  the point of contact of the last side. Then  $Y$  divides the line  $XZ$  in a constant ratio, and since  $Z$  describes a circle, therefore, etc.] \*

\* The following is an independent proof of the generalization of Weill's theorem.

Let  $ABCD \dots$  and  $A'B'C'D' \dots$  be any two positions of the variable polygon ;  $T_1, T_2, T_3, T'_1, T'_2, T'_3 \dots$  points of contact of the sides  $AB, BC, \dots$  ;  $A'B', B'C', \dots$  with the corresponding circles  $O_1, r_1, \delta_1$  ;  $O_2, r_2, \delta_2, \dots$  of the system ;  $R$  the point of intersection of  $AB$  and  $A'B'$  and  $\theta$  the angle between them ;  $S$  the intersection of  $AA'$  and  $BB'$ , and  $\phi$  the angle between them. Then  $AA', BB', CC' \dots$  touch a circle  $(\Omega, \rho, \lambda)$  coaxal with the given system. Let  $L, M, N \dots$  be its points of contact with  $AA', BB', CC'$ , etc. ... and we have

$$\frac{T_1 T'_1}{LM} = \frac{r_1 \sin \frac{1}{2} \theta}{\rho \sin \frac{1}{2} \phi} = \frac{r_1}{\rho} \cdot \frac{BM}{BT_1} = \frac{r_2}{\rho} \cdot \frac{\sqrt{\lambda}}{\sqrt{\delta_1}},$$

therefore  $\frac{\sqrt{\delta_1}}{r_1} \cdot T_1 T'_1 / LM = \frac{\sqrt{\delta_2}}{r_2} \cdot T_2 T'_2 / MN = \text{etc.}$

13. If the diagonals of a cyclic quadrilateral are conjugate lines and a homothetic quadrilateral be described with their intersection as homothetic centre ; prove that the consecutive pairs of sides of the one quadrilateral intersect the corresponding pairs of the other in eight points which lie on a circle coaxal with the circum-circles of the quadrilaterals. See Art. 96.

[Use the theorem of Art. 92, Ex. 2.]

---

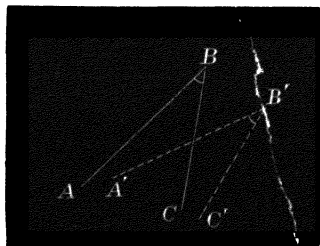
*i.e.*, multiples  $\sqrt{\delta_1}/r_1$ ,  $\sqrt{\delta_2}/r_2$ ,  $\sqrt{\delta_3}/r_3$  of the displacements  $T_1T_1'$ ,  $T_2T_2'$  ... are proportional to the sides of the polygon ; therefore, etc. Bowesman.]

## CHAPTER IX.

### SECTION I.

#### TWO SIMILAR FIGURES.

96. Two figures similar and similarly placed are said to be *Homothetic*, and their homologous parts are called *Corresponding Points, Lines*, etc. It is plain, if a line of either figure is displaced through an angle  $\theta$ , that every line of it is displaced through the same angle. For let  $AB$  be displaced to  $A'B'$ . It follows (Euc. III. 21, 22), since  $B=B'$ , that the angle between  $BC$  and  $B'C'$  is equal to  $\theta$ .



Also, since corresponding lines meet at equal angles, a variable pair of corresponding lines passing through a pair of corresponding points  $A$  and  $A'$  intersect on the circumference of a circle described on  $AA'$  containing an angle  $\theta$ ; and conversely.

Corresponding lines are made up of corresponding points; and the point of intersection of any two lines of either figure is the correspondent of the points of intersection of the corresponding lines of the other.

97. We have seen how to find a point  $S$  which, with the extremities of two linear segments  $AB$  and  $A'B'$ , forms similar triangles (Art. 25), and that it possesses the properties.

$\alpha^\circ$ . A variable line  $XX'$  dividing the segments similarly  $AX : BX = A'X' : B'X'$  subtends a constant angle at it; and

$\beta^\circ$ . Its distances from the lines are proportional to their lengths (Euc. VI. 19).

Now, if similar polygons be similarly described on  $AB$  and  $A'B'$ , it follows, as in Euc. VI. 20, that—

1°. The distances of  $S$  from each pair of corresponding lines are proportional to these lines.

2°. All pairs of corresponding points  $P$  and  $P'$  of the polygons subtend the same angle at it, and with it form a triangle of constant species.

3°. The polygons can be made homothetic by the revolution of either around it ( $2^\circ$ ).

For this reason it is called the *Homothetic Centre* of the Polygons, or their *Centre of Similitude*.

The ratio of  $SP$  to  $SP'$  is the *Ratio of Similitude* of the figures.

98. Since to each point  $P$  of one figure corresponds a point  $P'$  of the other such that  $PSP'$  is a triangle of constant species, if  $P$  coincides with  $S$ ,  $P'$  also coincides with

it; and therefore  $S$  taken as a point of either figure is its own correspondent in the other.

Hence it is a *Double Point* of the polygons.

99. From these considerations we make the following inferences :—

I. If upon the lines joining a fixed point  $S$  to the vertices of any polygon  $F_1$  similar and similarly situated triangles are constructed, their vertices form a polygon  $F_2$  similar to the given one, and  $S$  is their double point.

II. If the lines joining corresponding points of two directly similar figures are divided in the same ratio, the points of section form a polygon similar to the given ones (H. Van Aubel).

III. If the vertices of a polygon, constant in species, move on curves of any nature, to each position of it there is a corresponding centre of similitude.

This is called the *Instantaneous Centre* for the position, and is such that the lines drawn from it to all points  $A, B, C \dots X$  of the figure make equal angles with the tangents at these points to their respective loci.

[This is seen by taking two indefinitely near positions of the polygon.]

IV. Reciprocally :—If the lines  $L, M, N$  of the figure, moving as in the previous case. envelope curves, the lines joining the contacts of any position to  $S$  make equal angles with  $L, M, N$ .

[For the points of contact are the intersections of two consecutive positions of the moveable figure and are therefore corresponding points.]

## SECTION II.

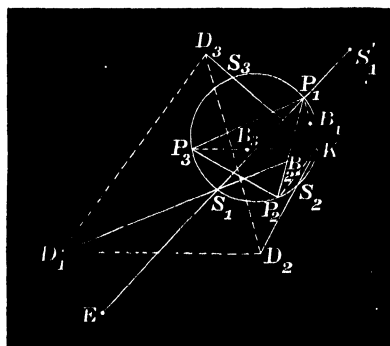
## THREE SIMILAR FIGURES.

100. Let  $F_1, F_2, F_3$  be any three directly similar figures;  $S_1$  the double point of  $F_2$  and  $F_3$ ;  $S_2$  and  $S_3$  the double points of the remaining pairs  $F_3, F_1$  and  $F_1, F_2$ ;  $a_1, a_2, a_3$  the lengths of corresponding lines  $d_1, d_2, d_3$ ;  $\alpha_1, \alpha_2, \alpha_3$  the angles of the triangle  $D_1D_2D_3$ , whose sides are  $d_1, d_2, d_3$ .

Then, by Art. 96,

1°. The variable triangle  $D_1D_2D_3$ , formed by any three corresponding lines, is constant in species.

2°. The distances of  $S_1$  from  $d_2$  and  $d_3$  are proportional to  $a_2$  and  $a_3$ , and similarly for  $S_2$  and  $S_3$  (Art. 97 ( $\beta^\circ$ )); therefore, the lines joining  $S_1, S_2, S_3$  to the *corresponding* vertices of  $D_1D_2D_3$  divide the angles  $D_1, D_2, D_3$  each into constant parts, and are concurrent (Art. 65).



Hence the triangle  $S_1S_2S_3$ , whose vertices are the centres of similitude of  $F_1, F_2, F_3$  taken in pairs (Triangle of

*Similitude*), is in perspective with all homologous triads  $D_1D_2D_3$ , etc.; and the centre of perspective  $K$  is a point such that its distances from any triad of homologous lines are in the ratios  $a_1 : a_2 : a_3$ .

3°. Since the base angles of each of the triangles  $D_2D_3K$ ,  $D_3D_1K$ ,  $D_1D_2K$  are constant (Art. 100, 2°) as  $D_1$ ,  $D_2$ ,  $D_3$  vary, the angles subtended by the sides of  $S_1S_2S_3$  at  $K$  are each constant, and the locus of  $K$  is therefore the circum-circle; hence,

*Any triangle formed by three homologous lines is in perspective with  $S_1S_2S_3$  at a point on the circum-circle of the latter; or the locus of the centre of perspective of  $S_1S_2S_3$  and any triangle formed by three homologous lines is the circum-circle of the former.* This is called the *Circle of Similitude* of  $F_1$ ,  $F_2$ ,  $F_3$ .

4°. The chords  $KP_1$ ,  $KP_2$ ,  $KP_3$  drawn parallel to  $d_1$ ,  $d_2$ ,  $d_3$  are homologous lines, for they intersect at angles  $\alpha$ ,  $\beta$ ,  $\gamma$ , and their distances from  $d_1$ ,  $d_2$ ,  $d_3$  are in the ratios  $a_1 : a_2 : a_3$ .\* Moreover, they meet the circle in fixed points, since the angle  $S_1KP_1$  is constant and  $S_1$  a fixed point; therefore  $P_1$  is fixed, and similarly  $P_2$  and  $P_3$  are fixed points.

They are termed the *Invariable Points*, and  $P_1P_2P_3$  the *Invariable Triangle*, of  $F_1$ ,  $F_2$ ,  $F_3$ .

4°. May be enunciated as follows:—

*All concurrent triads of homologous lines pass through the invariable points and intersect on the circle of similitude; and reciprocally:—the lines joining  $P_1$ ,  $P_2$ ,  $P_3$  to any three homologous points  $B_1$ ,  $B_2$ ,  $B_3$  meet in a point on the*

---

\* These lines are therefore the sides of an evanescent triangle  $D_1D_2D_3$  of constant species.

circle of similitude; and all triangles whose vertices are three homologous points are in perspective with  $P_1P_2P_3$  and the locus of their centre of perspective is the circle of similitude.

**101. Theorem.**—*The triangle of similitude and the invariable triangle are in perspective; and the distances of the centre of perspective  $E$  from the sides of the latter are inversely, as the ratios  $a_1 : a_2 : a_3$ .*

Since  $S_1$  is its own correspondent with respect to  $F_2$  and  $F_3$ ,  $P_2S_1$  and  $P_3S_1$  are homologous lines and lengths of these figures, therefore

$$S_1P_2 : S_1P_3 = a_2 : a_3, \dots\dots\dots(1)$$

but (Euc. III. 22)  $S_1P_2 : S_1P_3$  as the distances of  $S_1$  from  $P_1P_2$  and  $P_1P_3 = a_2 : a_3$  by (1), with similar relations for the points  $S_2$  and  $S_3$ ; therefore, etc., Art. 65.

**102. Theorem.**—*The invariable triangle is inversely similar to  $D_1D_2D_3$ .*

Follows by Euc. III. 22.

**103. Adjoint Points.\***—Let  $S'_1$  be the point of  $F_1$  which corresponds to  $S_1$  of the figures  $F_2$  and  $F_3$ .

Then  $S'_1S_1S_1$  is a particular case of a triangle formed by three homologous points, and is therefore (Art. 100, 4°) in perspective with  $P_1P_2P_3$  at a point on the circle of similitude; hence the lines  $P_1S'_1$ ,  $P_2S_1$ ,  $P_3S_1$  are concurrent. Their common point is therefore  $S_1$ ; that is to say,  $P_1S'_1$  passes through  $E$  and  $S_1$  (Art. 101); hence,

*The lines  $S_1S'_1$ ,  $S_2S'_2$ ,  $S_3S'_3$  meet each other in  $E$  and the circle of similitude at the invariable points.*

---

\* The theorems contained in Arts. 100-103 are due to Tarry. *Mathesis*, 1882, p. 72.



**Def.** The point  $E$  is called the *Director Point*, and  $S_1', S_2', S_3'$  the *Adjoint Points* of  $F_1, F_2, F_3$ .

104. **Theorems.\***—In any three similar figures there exists an infinite number of triads of homologous points  $C_1, C_2, C_3$  which are collinear. 2°. The loci of these points are circles passing through  $E$ . 3°. The variable line  $C_1C_2C_3$  turns around  $E$ . Neuberg.

The triangles  $S_1C_2C_3, S_2C_3C_1, S_3C_1C_2$  are constant in species (Art. 97, 2°); hence the angles  $S_2C_1S_3, S_3C_2S_1, S_1C_3S_2$  are given, and therefore the loci of the points are circles passing through each pair of double points.

Again, since  $S_2C_1C_2$  is a constant angle, the variable line  $C_1C_2$  meets the locus of  $C_1$  in a fixed point, and similarly it meets the loci of  $C_2$  and  $C_3$  in fixed points. Therefore the fixed points are coincident; that is to say, the circular loci have a point in common.

In the particular case of the collinear triads  $S_1'S_1S_2, S_2'S_2S_3, S_3'S_3S_1$  it has been proved (Art. 103) that their lines of collinearity pass through  $E$ ; therefore, etc. The points  $S_1', S_2', S_3'$  are on the corresponding circles.

105. **Particular Cases.**—Let the three similar figures  $F_1, F_2, F_3$  be described on the sides of a triangle  $ABC$ . It has been shown that the middle points of the symmedian chords of the circum-circle † are the common vertices of directly similar triangles described on the sides, taken in pairs (Art. 25, Ex. 2), and they are therefore the three double points. Hence,

1°. *Brocard's second triangle is the triangle of simili-*

\* *Mathesis*, 1882, pp. 76-8.

† The middle points of the symmedian chords of the circum-circle are the vertices of the triangle known as *Brocard's Second Triangle*.

tude, and the Brocard circle the circle of similitude, of three directly similar figures described on the sides of a triangle.

2°. Brocard's first triangle is their invariable triangle, Art. 29, Ex. 3.

3°. Brocard's second triangle and the given one are in perspective at a point on the circum-circle of the former whose distances from the sides of  $ABC$  are in the ratios of their lengths (Art. 100, 2°). See also Art. 16, Ex. 2.

4°. The centre of perspective is the symmedian point of  $ABC$ .

5°. The locus of the intersection of concurrent triads of homologous lines is the Brocard Circle, Art. 100, 4°.

6°. Brocard's first and second triangles are in perspective (Art. 101), and their centre of perspective  $E$ , or director points, is the centroid of  $ABC$ . (Art. 53, Ex. 6.)

7°. All collinear triads of homologous points lie on a variable line passing through  $E$ , and each point describes a circle passing through two vertices of Brocard's second triangle and the centroid of  $ABC$ .

#### M'CAY'S CIRCLES.

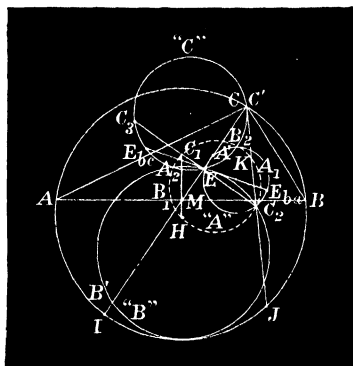
106. The loci in 7° of the previous Article are fully described by M'Cay in his memoir "On Three Circles related to a Triangle."\* Amongst many other properties they possess those given in this and the following Article.

The notation employed is as follows:— $ABC$  is the given triangle;  $A_1B_1C_1$ ,  $A_2B_2C_2$  Brocard's first and second triangles;  $E$  centroid;  $A'$ ,  $B'$ ,  $C'$  three homologous collinear points;  $M$  middle point of  $AB$ ;  $H$  circum-centre;

---

\* *Transactions of the Royal Irish Academy*, vol. xxviii.—Science.

$A_3, B_3, C_3$  the homologues of  $A_2, B_2, C_2$  respectively as double points of  $F_1, F_2, F_3$ .  $P_{ac}$  the  $c$  correspondent of  $P$  regarded as an  $a$  point, and  $L_{ac}$  and  $L_{ab}$  the  $c$  and  $b$



correspondents of any line  $L$  regarded as an  $a$  line; the circular loci the "A," "B," and "C" circles of the triangle.

1°. The mean centre of any three collinear homologous points is at  $E$  (Art. 53, Ex. 6).

2°. If one of them  $C'$  coincides with  $E$ ,  $A'B'$  is a tangent to the "C" circle and  $EA' = EB'$  or  $EE_{ca} = EE_{cb}$ ; similarly we have  $EE_{ab} = EE_{ac}$  and  $EE_{bc} = EE_{ba}$ .

3°. If one of them coincides with a double point  $A_2$ , the line of collinearity is  $A_2EA_1A_3$  (Art. 103) and  $EA_3 = 2EA_2$ .

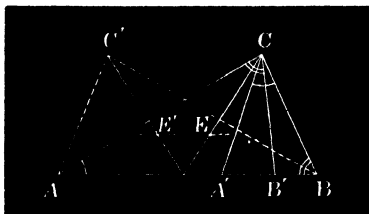
Similarly the lines  $B_1B_2B_3$  and  $C_1C_2C_3$  each pass through  $E$ , which is the common point of trisection of the segments  $A_2A_3, B_2B_3, C_2C_3$ .

4°. The circles cut each other at angles  $A, B$ , and  $C$ .

5°. Their centres are on the perpendicular bisectors of the sides.

This is proved for the "C" circle as follows:—

On the sides of  $ABC$  construct three directly similar triangles  $BCA'$ ,  $CAB'$ ,  $ABC'$ , each inversely similar to



$ABC$ . Their centres of gravity are therefore corresponding points. But they lie on a parallel through  $E$  to  $AB$ ; hence  $E'$ , the centroid of  $ABC'$ , is on the "C" circle and  $E$  and  $E'$  are reflexions with respect to the perpendicular bisector of  $AB$ .

**107. Problem.**—*To find the Centres and Radii of M'Cay's Circles.*

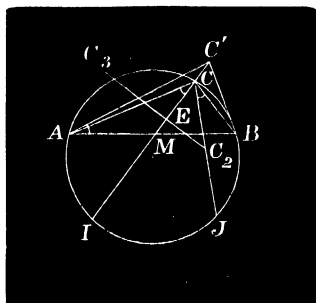
This is done by finding where the circles again cut the corresponding medians. We take, for example, the "C" circle and require to find  $C'$ . Let  $L$  denote the median  $CM$ , and take it an  $a$  line. Since it makes an angle  $BCM$  with the side  $a$ , we draw the corresponding  $b$  and  $c$  lines by making angles  $CAB'$  and  $ABC'$  equal to  $BCM$ .

From similar triangles  $MBC'$  and  $MCB$  we have  $MC \cdot MC' = MB^2 = MC \cdot MI$ ; hence  $MC' = MI$ . This also follows, since the triangles  $ABI$  and  $BAC'$  are similar.

Again, the triangle  $CBC_2$  is inversely similar to  $ABC'$ , but it is (hyp.) directly similar to  $BAC_3$ . Hence  $BAC_3$  and  $ABC'$  are inversely similar; therefore  $C'$  is the

reflexion of  $C_3$  with respect to the perpendicular bisector of the base.

The connection between three collinear points  $A', B', C'$  on the median to the side  $c$  of the given triangle and  $C_2, C_2', C_3$  has thus been established.



The triangles  $BCA', CAB', ABC'$  are similar to one another, and to  $CBC_2, ACC_2$ , and  $BAC_3$ ; and therefore  $A', C_2; B', C_2; C', C_3$  are reflexions of one another with respect to the corresponding perpendicular bisectors of the sides of  $ABC$ .

It follows that if the median and symmedian cut the circum-circle in  $I$  and  $J$ , and these points be joined to  $M$ , the lines  $MI$  and  $MJ$  produced through  $M$  pass through  $C'$  and  $C_3$  respectively;  $MJ = MC_3$  and  $MI = MC'$ , or  $C'$  and  $C_3$  are the reflexions of  $I$  and  $J$  with respect to the base  $AB$ .

Let  $d$  be the distance of the centre of the " $C$ " circle from  $AB$ ,  $m$  the median, and  $\theta$  the angle it makes with the base,  $t$  the tangent from  $M$  to the circle. Then

$$t^2 = ME \cdot MC' = ME \cdot MI = \frac{c^2}{12} \dots \dots \dots (1)$$

$$\text{Again, } 2d \sin \theta = ME + MC' = \frac{m}{3} + \frac{3t^2}{m} = \frac{a^2 + b^2 + c^2}{6m} \dots (2)$$

by (1); whence  $d = \frac{1}{3}c \cot \omega$ , and the radius of the "C" circle is given by the equation

$$\rho = \sqrt{d^2 - t^2} = \frac{1}{3}c \sqrt{\cot^2 \omega - 3} \text{ (cf. Art. 28, Ex. 19).}$$

Also, since the highest and lowest points of the circle are distant from the base  $\rho + d$  and  $\rho - d$ , these quantities are the roots of the quadratic equation

$$12h^2 - 4c \cot \omega \cdot h + c^2 = 0; \dots \dots \dots (3)$$

or, putting  $h = \frac{1}{2}c \tan \phi$ ,

$$3 \tan^2 \phi - 2 \cot \omega \cdot \tan \phi + 1 = 0, \dots \dots \dots (4)$$

an equation which reduces by an easy transformation to

$$\sin(\omega + 2\phi) = 2 \sin \omega \dots \dots \dots (5)$$

The forms (4) and (5) are remarkable inasmuch as they express  $\phi$  as a symmetric function of the angles; hence,

*Three similar isosceles triangles may be constructed on the sides of ABC, whose vertices are a triad of collinear homologous points.*

Let  $P, Q, R$  be the vertices of these triangles. Since

$$HR = HM - MR = R \cos A - \frac{1}{2}a \tan \phi = \frac{R \cos(A + \phi)}{\cos \phi},$$

with similar values for  $HP$  and  $HQ$ ; also, from the collinearity of  $P, Q, R$  we have  $\Sigma \frac{\sin A}{HP} = 0$ .

By substitution, we obtain

$$\frac{\sin A}{\cos(A + \phi)} + \frac{\sin B}{\cos(B + \phi)} + \frac{\sin C}{\cos(C + \phi)} = 0, \dots \dots \dots (6)$$

an equation which is therefore identical with the forms (4) and (5).

Let  $h_1$  and  $h_2$  be the roots of (3), then

$$\frac{1}{h_1} + \frac{1}{h_2} = \frac{4 \cot \omega}{c} = \frac{2}{\frac{1}{2}c \tan \omega} = \frac{2}{MC'},$$

where  $C'$  is the vertex of Brocard's first triangle; therefore

*The vertices of Brocard's first triangle and the corresponding sides of  $ABC$  are pole and polar with respect to the "A," "B," and "C" circles.*

Many other beautiful properties of these circles are given in the memoir from which the preceding are extracts.

108. If  $A', B', C'$  be the feet of the perpendiculars of  $ABC$ , the triangles  $AB'C'$ ,  $A'BC'$ , and  $A'B'C$  are similar, and may therefore be taken as portions of three directly similar figures  $F_1, F_2, F_3$ , whose double points are  $A', B', C'$ , homologous lines in the ratios  $\cos A : \cos B : \cos C$ , the middle points of the segments of the perpendiculars towards the angles  $A'', B'', C''$ , the invariable points  $A''', B''', C'''$ , points of concurrence of homologous lines middle points of sides, and the nine-points-circle the circle of similitude (Neuberg).

#### EXAMPLES.

1. If similar figures  $F_1, F_2, F_3$  be described on the perpendiculars  $AA', BB', CC'$  of a triangle, their circle of similitude is the ortho-centroidal circle.

[For the orthocentre being the point of concurrence of three corresponding lines is on the circle of similitude (Art. 100, 4°). Also the parallels through the centroid  $E'$  to the sides of the triangle trisect the perpendiculars at right angles, and are therefore also corresponding lines; therefore, etc.]

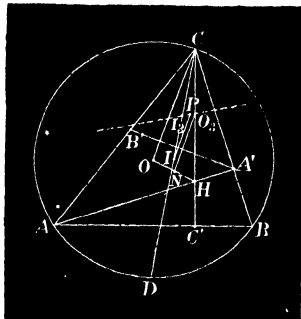
We note that the parallels meet the corresponding perpendiculars in  $P, Q, R$ , the invariable points of  $F_1, F_2, F_3$ .]

2. The lines joining the in- and circum-centres of the *copedal* triangles  $B'C'A, C'A'B, A'B'C$  meet at the point of contact of the nine-points and in-circle of  $ABC$ .

[By Art. 108, the three triangles being parts of similar figures have the nine-points-circle of  $ABC$  for circle of similitude, and the middle points of the segments of the perpendiculars for invariable points; hence (Art. 100, 4°), if  $I_1, I_2, I_3$  and  $O_1, O_2, O_3$  denote the in- and circum-centres of the triangles, the lines  $I_1O_1, I_2O_2, I_3O_3$  correspond, and are concurrent on the circle of similitude.

Dr. Casey \* proves the remainder of the property, which includes *Feuerbach's Theorem*, as follows:—

Let  $N$  be the nine-points-centre; then  $NO_3 = \frac{1}{2}R$ . Draw  $IP$  parallel to  $NO_3$ . Now, if  $PI$  is proved to be equal to the radius of the in-circle, the line  $I_3O_3$  is the join of parallel radii, and therefore passes through a centre of similitude of the circles; similarly for  $I_1O_1$  and  $I_2O_2$ .



Since  $COI$  and  $CO_3I_3$  are corresponding parts of similar figures, they are similar; therefore the angle  $DIO = I_3P$ , and  $ODI = OCI = CIP$ , since  $NO_3$  is parallel to  $OC$ . Hence the triangles  $ODI$  and  $PII_3$  are similar, and

$$\frac{IP}{R} = \frac{II_3}{ID} = \frac{II_3 \cdot IC}{2Rr} = \frac{2r^2}{2Rr} \left( = \frac{r}{R} \right),$$

since  $CI/CI_3 = 1/\cos C$ , the ratio of similitude of  $ABC$  and  $A'B'C'$  (Art. 108).]

3. If  $A$  and  $A'$ , corresponding points of two similar figures, are conjugate points with respect to a fixed circle, required to find their loci.

\* Casey's *Sequel to Euclid*, fifth edition, p. 202.



[Take  $S$  the double point,  $M$  the middle point of  $AA'$ . Then  $SAA'$  is a triangle of constant species; therefore  $SM/MA$  is a constant ratio. But  $MA=t$ , the tangent from  $M$  to the given circle (Art. 73, 2°). Hence  $SM/t$  is constant and  $M$  describes a circle (Art. 72, Ex. 3); therefore also  $A$  and  $A'$  describe circles.]

4. If  $X_1X_2X_3$  be a triangle formed by joining a triad of corresponding points of three similar figures such that  $X_1X_2:X_1X_3 = \text{const.}$ , the locus of each vertex is a circle.

[The triangle  $S_3X_1X_2$  is constant in species, Art. 97; similarly for  $S_2X_1X_3$ ; hence  $S_3X_1/X_1X_2$  and  $S_2X_1/X_1X_3$  are constant ratios. Dividing one by the other, we have the base  $S_2S_3$  and ratio of sides of the triangle  $S_2S_3X_1$ ; therefore, etc.]

It is to be noted that as the ratio  $S_2X_1/S_3X_1$  varies in magnitude the vertex  $X_1$  describes a coaxal system of which  $S_2$  and  $S_3$  are the limiting points.]

5. If the area of  $X_1X_2X_3$  is given, each vertex describes a circle.

[For  $X_1X_2 \cdot X_1X_3 \sin X_1$  varies as  $S_2X_1 \cdot S_3X_1 \sin(X_1 - \theta)$ ; therefore, etc. (Art. 23, Ex. 3).  $X_2$  and  $X_3$  similarly describe circles.]

6. If a side or an angle of  $X_1X_2X_3$  is given, its vertices describe circles.

7. If the area of a variable triangle formed by three corresponding lines be given, its sides envelope circles whose centres are the invariable points of  $F_1, F_2, F_3$ .

These and many other excellent illustrations of the theory of three directly similar figures are to be found in Casey's *Sequel to Euclid*, to which the student is referred. See fifth edition, Miscellaneous Examples, pp. 231-248.

## CHAPTER X.

### SECTION I.

#### CENTRES OF SIMILITUDE.

109. If  $A, r_1$ ;  $B, r_2$  be any two non-intersecting circles,  $P$  and  $Q$  the points of intersection of the direct and transverse common tangents, it is easily proved that  $A, B, P, Q$  are collinear, and that  $AP/BP = AQ/BQ = r_1/r_2$ ; hence the *centres of similitude of two circles are the points of intersection of the direct and transverse common tangents*.\*

In the case of intersecting circles, if  $C$  be a point of intersection, we infer from these equations that the bisectors of the angle between the circles meet the line of centres in  $P$  and  $Q$  (Euc. VI. 3).

For the in- and ex-circles of a triangle taken in pairs the twelve centres of similitude are the vertices and the points where the bisectors of the angles meet the opposite sides.

The centres of similitude of a line  $L$  and circle  $A$  are the extremities of the diameter perpendicular to  $L$ .

For the common tangents to the circle and line are

---

\* Therefore the common tangents, real or imaginary, to any two circles always intersect in real points.



*parallel.* Thus the tangents at  $A_2$  and  $B_2$  are parallel. More generally any two points  $A_n$  and  $B_n$  which connect through  $C$  such that  $CA_n/CB_n = r_1/r_2$  are homologous.  $A_1$  and  $B_2$  are termed *Antihomologous Points*, and since the radii  $AA_1$  and  $BB_2$  through them make equal angles with their line of connexion, *the tangents at antihomologous points meet on the radical axis.*

Let a second transversal through  $C$  meet the circles in  $A_3A_4B_3B_4$ . The chords  $A_1A_3$  and  $B_1B_3$  joining pairs of homologous points are termed *Homologous Lines*, and those joining pairs of antihomologous points *Antihomologous Lines*. Thus  $A_2A_4$ ,  $B_1B_3$ , and  $A_1A_3$ ,  $B_2B_4$  are pairs of antihomologous lines.

**111. Theorem.**—*Homologous chords ( $A_1A_3$ ,  $B_1B_3$ ) of any two circles are parallel.*

For it has been shown that  $AA_1$  and  $BB_1$ ,  $AA_2$  and  $BB_2$  are pairs of parallel lines; hence the two isosceles triangles  $AA_1A_3$  and  $BB_1B_3$  have equal vertical angles, and are therefore similar (Euc. VI. 6).

NOTE.—Since any line through  $C$  meets homologous lines  $A_1A_3$  and  $B_1B_3$  in homologous points  $A_n$  and  $B_n$ , therefore  $A_n$ ,  $B_n$  are in general the corresponding intersections of pairs of homologous lines. The two points  $A_1A_3$ ,  $A_2A_4$  and  $B_1B_3$ ,  $B_2B_4$  are homologous.

**112. Theorem.**—*Antihomologous chords ( $A_2A_4$ ,  $B_1B_3$ ) of any two circles meet on their radical axis.*

By Art. 111, we have  $CA_1/CA_3 = CB_1/CB_3$ , but (Euc. III. 36)  $CA_1/CA_3 = CA_4/CA_2$ ; hence  $CB_1/CB_3 = CA_4/CA_2$  or  $CA_2 \cdot CB_1 = CA_4 \cdot CB_3$ ; thus:—*any two points are concyclic with the corresponding pair of antihomologous points; therefore, etc. (Art. 88, Ex. 6).*

## PRODUCTS OF ANTISIMILITUDE.

113. By the previous Article, we have from the cyclic quadrilateral  $A_2A_4B_1B_3$

$$CA_2 \cdot CB_1 = CA_4 \cdot CB_3.$$

We may therefore infer that *the rectangle under the distances of either centre of similitude from a pair of antihomologous points is constant.*

If the circles  $A, r_1; B, r_2$  be regarded as portions of two geometrical figures, any point  $A_n$  of one is antihomologous to  $B_n$  of the other when the line  $A_nB_n$  passes through a centre of similitude  $C$ , and  $CA_n \cdot CB_n$  is equal to the above constant, which is termed the *Product of Antisimilitude (External or Internal)*.

**To find the values of the products,** we take the extreme positions of the variable line  $CA_2B_1$  which for real intersections are the common tangents.

We have therefore

$$CA_2 \cdot CB_1 = CT_1 \cdot CT_2 \dots \dots \dots (1)$$

Again, since  $T_1T_2$  subtends a right angle at each of the limiting points  $M$  and  $N$  (Art. 88, Cor. 3),

$$CT_1 \cdot CT_2 = CM \cdot CN \dots \dots \dots (2)$$

These constant values which may be expressed in terms of the distance ( $\delta$ ) between the centres of the given circles and their radii ( $r_1$  and  $r_2$ ) are of importance in the theory of coaxal circles, and will frequently be made use of in the next chapter.

Join  $AT_1$  and  $BT_2$ . Let  $ACT_1 = \theta$ .

$$\begin{aligned} \text{Then } CT_1 \cdot CT_2 &= r_1 r_2 \cot^2 \theta = r_1 r_2 \cdot \left( \frac{T_1 T_2}{r_2 - r_1} \right)^2 \\ &= \frac{r_1 r_2}{(r_1 - r_2)^2} [\delta^2 - (r_1 - r_2)^2] \dots \dots \dots (3) \end{aligned}$$

Similarly the internal product of antisimilitude is found to be equal to

$$\frac{r_1 r_2}{(r_1 + r_2)^2} [(r_1 + r_2)^2 - \delta^2] \dots \dots \dots (4)$$

NOTE.—It should be noticed when the two circles lie wholly outside each other  $\delta > r_1 + r_2$ , if they intersect  $\delta < r_1 + r_2$  and  $> r_1 \sim r_2$  (Euc. I. 20), and when one lies completely within the other  $\delta < r_1 \sim r_2$  (Euc. III. 12); hence it follows from (3) that the external product of antisimilitude is *negative* only when one circle lies wholly within the other. Also from (4) the internal product is *negative* when the circles are external to one another and *positive* in every other case. In the case where both products are positive  $\delta > r_1 \sim r_2$  and  $< r_1 + r_2$ ; therefore  $\delta, r_1, r_2$  form a triangle (Euc. I. 20), or the circles intersect in a pair of real points.

# EXAMPLES.

1. If a variable circle touch two circles with contacts of similar species, its points of contact are antihomologous points.

[By Art. 112, if  $AA_2$  and  $BB_1$  be produced to meet in  $X$ ,  $XB_1 = XA_2$ . In the case of internal contact the points of contact are  $A_1, B_2$ .]

2. Describe a circle passing through a given point ( $P$ ) and touching two fixed circles ( $A, r_1$ ) ( $B, r_2$ ).

[By Art. 110, the required circle passes through an antihomologous point  $P'$ , and the problem thus reduces to “describe a circle passing through two fixed points and touching a given circle.”]

3. The polars of the external centre of similitude with respect to two circles are equidistant from the radical axis, and therefore also from the limiting points.

4. The line at infinity is an axis of perspective of two circles.

[Regard the circles as similar polygons of an infinite number of sides, and join their corresponding vertices (*i.e.* the homologous points). Thus the ex-centre of similitude is a *Centre of Perspective* of the circles. Again, the corresponding sides (*i.e.* homologous lines) intersect on the axis of perspective. In this case they are parallel.

Hence *the line at infinity is the axis of perspective of every two circles.* (Cf. Art. 87).]

5. The radical axis is also an axis of perspective of two circles.

[For since antihomologous points  $B_1, A_2$  connect through a centre of similitude  $C$ , the circles may be regarded as polygons of an infinite number of sides whose corresponding vertices are antihomologous points and whose corresponding sides are therefore antihomologous lines; but these latter intersect on the radical axis (Art. 112), which is therefore the axis of perspective.\*

6. The poles  $A_n, B_n$  of the chords  $A_1A_2$  and  $B_1B_2$  are homologous points.

[For they are the intersections of pairs of homologous lines, viz. the tangents at  $A_1, A_2$  and  $B_1, B_2$  respectively.]

7. In Ex. 6 the lines  $A_1B_1$  and  $A_nB_n$  are conjugate with respect to both circles.

8. If  $C, C'$  denote the centres of similitude of two circles which cut orthogonally at  $X$ ; the inverse ( $C''$ ) of the point  $C'$  with respect to the circle  $A$  is the inverse of  $C$  with respect to the circle  $B$ .

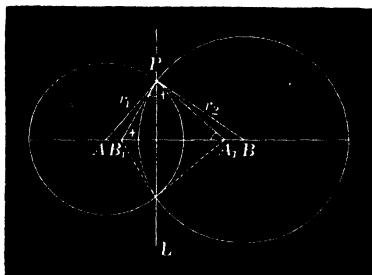
[Since  $C'$  and  $C''$  are inverse points,  $AC''X = AXC' = 45^\circ$ ; hence  $AC''X = BXC$ , therefore  $CB/BX = BX/BC''$ , therefore etc.

9. A variable circle touches two equal circles with contacts of opposite species: show that the product of the intercepts on their transverse common tangents made by the perpendiculars from the centre and measured from their point of intersection is constant.

10. The centres of similitude, the centre of the circle of similitude, and the centre of either circle  $B$  are pairs of inverse points with respect to a circle concentric with  $A$ .

\* Two circles are thus shown to be doubly in perspective to each centre of similitude; the two axes of perspective forming the coaxal circle whose radius is infinitely great, viz., the radical axis and the line at infinity. It follows that "for every two circles in the same plane, however circumstanced as to magnitude and position, the radical axis and the line at infinity, being both axes of perspective, are both chords of intersection; the corresponding points of intersection, real or imaginary, according to circumstances in the case of the former, being of course from the nature of the figures always imaginary in the case of the latter." (Townsend.)

11. The poles  $A_1, B_1$  of the radical axis of two circles  $(A, r_1; B, r_2)$  are inverse points with respect to their circle of similitude.



[For since  $AA_1 \cdot AL = r_1^2$ , angle  $APL = AA_1P$ ;  
also since  $BB_1 \cdot BL = r_2^2$ , angle  $BPL = BB_1P$ .  
By addition  $APB = PA_1B_1 + PB_1A_1 = \pi - A_1PB_1$ .

Thus  $A_1, B_1$  and  $A, B$ , since they subtend similar angles at  $P$ , are pairs of inverse points with respect to the circle of similitude (Art. 72, Cor 8).]

12. If a variable circle  $V$  cut two circles  $A$  and  $B$  at constant angles, show that the centre of similitude of any two positions  $V_1$  and  $V_2$  is on  $L$  the radical axis of  $A$  and  $B$ .

[For  $V_1$  and  $V_2$  meet the line  $L$  at equal angles (Art. 88, Ex. 7); therefore it passes through their ex-centre of similitude.]

12a. Hence show that if the circles  $A$  and  $B$  each cut three fixed circles  $V_1, V_2, V_3$  at the same angles  $\alpha, \beta, \gamma$ , an axis of similitude of the three is the radical axis of the two.

13. Construct a quadrilateral, having given the four sides, and that two adjacent angles are equal. (*Mathesis*, 1881.)

14. **Feuerbach's Theorem.** To prove by an elementary method that the nine-points-circle touches the in-circle.

Draw  $C'X$  the fourth common tangent to the in- and ex-circles to the side  $c$  of the triangle  $ABC$ . We shall prove that the line joining  $M$ , the middle point of the base, to the point of contact  $X$  passes through the point of contact  $Y$  of the in- and nine-points-circles.

Let  $T$  be the point of contact of the in-circle,  $P$  the foot of the perpendicular, and  $C'$  the foot of the internal bisector of  $C$ .





## SECTION II.

## CIRCLES OF ANTISIMILITUDE.

**Definitions.** The circle described with either centre of similitude of two given circles as centre, the square of whose radius is equal to the corresponding product (Art. 113) of antisimilitude, is known as a *Circle of Antisimilitude*.

Thus there are two circles of antisimilitude, *External* and *Internal*, according as the centre coincides with the external or internal centre of similitude of the given circles.

From the definition it is evident that all pairs of antihomologous points are *inverse points with respect to the circle of antisimilitude*, or, more generally, that *each of the two given circles is the inverse of the other with respect to either circle of antisimilitude*.

In the next chapter this latter circle, from this fundamental property, will be otherwise known as the *Circle of Inversion* of the two given ones.

114. The following theorems are of importance in the geometry of these circles.

1°. Any two circles  $A$  and  $B$  and their circles of antisimilitude are coaxal.

For the constant product  $CA_2 \cdot CB_1$  (Art. 113) has been proved equal to  $CM \cdot CN$ ; hence  $M$  and  $N$  are a common pair of inverse points to the four circles.

2°. The squares of the tangents  $t_1$  and  $t_2$  from any point of either circle of antisimilitude to  $A$  and  $B$  are in the ratio of the radii; or  $t_1^2 : t_2^2 = r_1 : r_2$ .



EXAMPLES.

1. A variable circle passing through a fixed point and cutting two given ones at equal angles passes through a second fixed point.

[In every position it passes through the inverse of the fixed point with respect to the ex-circle of antisimilitude.]

2. A variable circle passing through a fixed point and cutting two fixed circles at supplemental angles passes through a second fixed point.

[The inverse of the given one with respect to the in-circle of antisimilitude.]

3. Two circles  $X$ ,  $Y$  intersecting two others  $A$  and  $B$  at equal angles have for radical axis a line passing through the centre  $C$  of the ex-circle of antisimilitude of  $A$  and  $B$ .

[For if  $X$  and  $Y$  intersect in a point  $P$ , each must pass through the inverse of  $P$  with respect to  $C$ .]

3a. If the angles are supplemental, the radical axis of  $X$  and  $Y$  passes through the in-centre of antisimilitude.

4. If three circles  $X$ ,  $Y$ ,  $Z$  meet two others  $A$  and  $B$  at equal or supplemental angles, the radical centre of the three coincides with the external or internal centre of similitude  $C'$  or  $C''$  of the two.

[For by Ex. 3 the radical axes of  $Y$ ,  $Z$ ;  $Z$ ,  $X$ ;  $X$ ,  $Y$  each pass through  $C'$  or  $C''$  according as the angles of section are equal or supplemental; therefore, etc.]

NOTE.—In this example it may be noticed that in the first case the circles  $A$  and  $B$  each cut  $X$ ,  $Y$ , and  $Z$  at equal angles; therefore they cut the ex-circles of antisimilitude of  $Y$ ,  $Z$ ;  $Z$ ,  $X$ ;  $X$ ,  $Y$  at right angles (Art. 114). But the ex-circles of antisimilitude are coaxal; hence *a variable circle  $A$  cutting three others  $X$ ,  $Y$ ,  $Z$  at equal angles describes a coaxal system, the conjugate of that formed by the circles of antisimilitude of  $X$ ,  $Y$ ,  $Z$  taken two and two.* More generally, a variable circle cutting three others  $X$ ,  $Y$ ,  $Z$  at similar angles describes four coaxal systems whose radical axes are the four axes of similitude of  $X$ ,  $Y$ ,  $Z$ . Also, since the common ortho-

gonal circle of the three cuts them at once at equal and supplemental angles, it belongs to each of the four coaxal systems.

5. If two circles  $A$  and  $B$  touch with similar contacts three others  $X$ ,  $Y$ ,  $Z$ , the radical axis of  $A$  and  $B$  is the line joining the ex-centres of similitude of  $X$ ,  $Y$ ,  $Z$  taken in pairs.

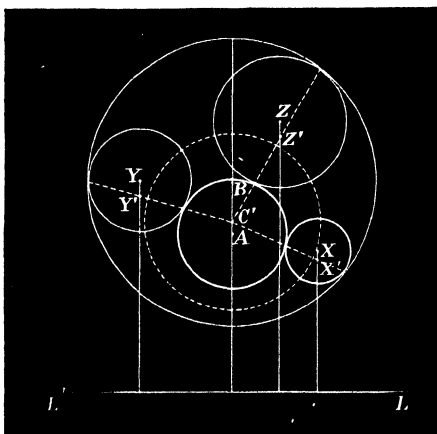
[A particular case of the foregoing.]

6. The eight circles that can be described to touch three given ones arrange themselves in pairs coaxal with the four axes of similitude of the given ones.

7. In Ex. 5 the chords of the three circles joining the points of contact with the two meet at the in-centre of similitude of  $A$  and  $B$ , and therefore at the radical centre of  $X$ ,  $Y$ ,  $Z$ .

8. The chords of contact pass through the poles of the radical axis of  $A$  and  $B$  with respect to each of the circles  $X$ ,  $Y$ ,  $Z$ .

[For the tangents at the extremities of the chord of contact of  $X$  being equal intersect on the radical axis of  $A$  and  $B$ .]



NOTE.—Gergonne deduces by means of the foregoing properties a simple geometrical construction for the eight circles of contact of

three given ones  $X, Y, Z$ . The circles having similar contacts are found as follows:—Find the ex-centres of similitude of  $X, Y, Z$  taken in pairs; the line  $L$  joining them is the radical axis of the required circles  $A$  and  $B$ . Next find  $C'$  the centre of the common orthogonal circle of the given ones.  $C'$  is the in-centre of similitude of  $A$  and  $B$ . Now obtain the inverses  $X', Y', Z'$  of  $L$  with respect to  $X, Y, Z$  respectively. Join  $C'X', C'Y'$ , and  $C'Z'$ ; these lines meet the given circles at the required points of contact; therefore, etc. The remaining circles may be similarly found.

Otherwise, thus:—By Casey's relation in Art. 7, if we number the given circles 1, 2, 3 and let 4 be the required point of contact with 1, we have the ratio of the tangents from 4 to 2 and 3, a given quantity  $k$ . Similarly for the second circle which has the similar contacts with the three given ones, the ratio of the tangents from its point of contact (5) to 2 and 3 = the same ratio  $k$ ; therefore, etc. (Art. 88, Cor. 1).

9. Let  $A_1A_2, B_1B_2$  be the extremities of the common diameter of two circles;  $M, N$  their limiting points; prove that the circles on  $A_1B_1, A_2B_2, MN$  as diameters are coaxal.

[For their centres are collinear, and they each cut the internal circle of antisimilitude orthogonally (Art. 114, 4°); therefore, etc.]

10. A variable circle cutting three given ones at equal angles passes through two fixed points, real or imaginary.

[For it cuts the external circles of antisimilitude of the given ones taken two and two orthogonally, and these (Art. 88, Ex. 13. 2°) are coaxal; therefore the variable circle passes through their limiting points, real or imaginary.]

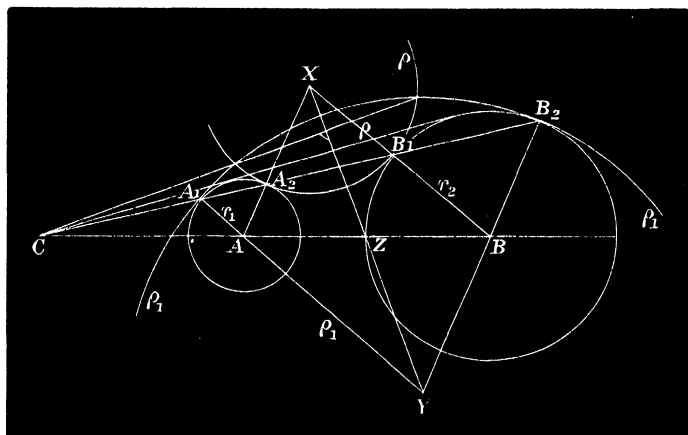
11. Two variable circles  $X$  and  $Y$  touch externally two fixed circles  $A, r_1$  and  $B, r_2$  at four points  $B_1, A_2$  and  $A_2, B_1$  in a right line; prove that

$\alpha^\circ$ . The line joining their centres passes through a fixed point.

$\beta^\circ$ . The sum of their radii is constant.

$\gamma^\circ$ . The foot of their radical axis describes a circle.

[ $\alpha^\circ$ . Since the diagonals of a parallelogram bisect each other,  $XY$  bisects and is bisected at the middle point  $Z$  of  $AB$ .



$\beta^\circ$ . Let  $L$  be the radical axis of  $A, r_1$ , and  $B, r_2$ ; then  $XL/\rho = YL/\rho_1 = \text{const.}$  (Art. 88, Ex. 7), and therefore  $\frac{XL + YL}{\rho + \rho_1} = \text{const.}$ , but the numerator is constant by  $\alpha^\circ$  ( $= 2ZL$ ); therefore, etc.

$\gamma^\circ$ . The circle on  $CZ$  is evidently the locus.]

12. Circles are described touching two fixed circles (as in Fig. of Ex. 8); find the locus of the limiting points of these circles taken in pairs.

[The internal circle of antisimilitude of the two given circles (Art. 114, 3 $^\circ$ ).]

12a. Circles are described touching one another, and each touching two given circles; find the locus of their points of contact.

[The points of contact are the coincident limiting points of the touching circles; hence the required locus is the internal circle of antisimilitude of the two given ones.]

13. If  $n$  points be taken on a circle, prove that (1) the mean centres of the  $n$  systems of  $n-1$  points formed by omitting each

point in succession, lie on a circle  $S_n$ ; (2) if another point be taken on the original circle, the centres of the  $n+1$  circles ( $S_n$ ) obtained by omitting each point in succession lie on an equal circle; and so on *ad infinitum*. (*St. Clair.*)\*

[Let  $G$  be the mean centre of the system of  $n$  points. Produce  $AG$  to  $a$ , making  $AG : Ga = n-1 : 1$ ; then  $a$  is the mean centre of the  $n-1$  points formed by excluding  $A$ . In the same manner we get  $BG : Gb = n-1 : 1$ , etc.; hence the points  $a, b, \dots$  lie on a circle; and  $G$  is a centre of similitude of the locus circle and the given one.

---

\* *Educational Times*, February, 1891.



## CHAPTER XI.

### INVERSION.

#### SECTION I.

##### INTRODUCTORY.

115. It has been seen (Art. 74) that the inverse of every point on a line with respect to a circle lies on a circle described on the line joining the centre of the given circle with the pole of the line.

This circle is said to be the inverse of the line with respect to the given circle; and it may be generally inferred that *the inverse of a line is a circle passing through the centre of the given circle*; and conversely. This latter is named the *Circle of Inversion*, and its centre the *Origin* or *Centre of Inversion*.

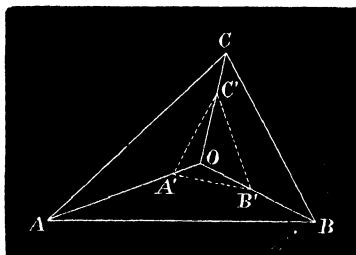
We shall now proceed to discuss the inversion of a system of points which are not collinear. Take the simplest case—the vertices of a triangle  $ABC$ . Let their inverses with respect to a circle of inversion  $O, r$  be respectively  $A', B', C'$ .

It is obvious that the three quadrilaterals  $BCB'C'$ ,  $CAC'A'$ ,  $ABA'B'$  are cyclic; hence we have the angular relations:—

$$A'C'O = OAC, B'C'O = OBC, \text{ etc. (Euc. III. 22),}$$

and thence by addition,

$$AOB = C + C' \dots\dots\dots(1)$$



$$\text{Similarly,} \quad BOC = A + A' \dots\dots\dots(2)$$

$$\text{and} \quad COA = B + B' \dots\dots\dots(3)$$

If the base  $AB$  and origin  $O$  are fixed, and  $C$  given in magnitude,  $C'$  is also given in magnitude by (1); hence :  
*—If a variable point ( $C$ ) describes a circle (circum-circle of  $ABC$ ), the locus of its inverse ( $C'$ ) is a circle ( $A'B'C'$ ).\**

Two circles or, more generally, any two curves so related that every point of one has a *Corresponding Point* on the other inverse to it with respect to a given circle, are *Inverse Figures* with respect to the circle of inversion.

It has thus been proved that in general a line or circle

\* This statement is equivalent to the following :—

*—If a variable line  $OPP'$  is drawn from a fixed point  $O$  to a given circle and divided at  $X$  such that  $OP \cdot OX = \text{const.}$ ; the locus of  $X$  is a circle, which may be thus proved independently. Since  $OP \cdot OP'$  and  $OP \cdot OX$  are both constant,  $OX : OP' = \text{const.}$  Through  $X$  draw  $XC'$  parallel to  $CP'$ . From similar triangles  $OX : OP' = OC' : OC = C'X : CP' = \text{const.}$  Hence  $C'$  is a fixed point, and  $C'X$  is of constant length. The locus of  $X$  is therefore a known circle; and the circle of inversion is obviously a circle of antisimilitude of the given one and its inverse.*

inverts into a circle; and in the particular case when the origin is on the circle, its inverse is a line.

**116. Species of  $A'B'C'$ .** Let the points  $A, B, C$  be fixed. Since  $AOB = C + C'$ ,  $C'$  may have any value depending on the position of the point  $O$ . The following particular cases are worthy of notice, and may be readily inferred:—

1°. If  $O$  is the circum-centre of  $ABC$ ,

$$A = A', B = B', C = C'.$$

2°. If  $O$  is the right (or positive) Brocard point of  $ABC$ ,

$$AOB = C + C' = \pi - B;$$

hence

$$C' = A.$$

Similarly

$$A' = B \text{ and } B' = C.$$

3°. If  $O$  is the left (or negative) Brocard point,  $ABC$  and  $A'B'C'$  are again similar.

4°. If  $O$  is one of the vertices ( $C_2$ ) of Brocard's second triangle,  $AOB = 2C = C + C'$ , therefore  $C = C'$ ; and also  $B = A'$  and  $A = B'$ .

Hence the triangles are similar when the centre of inversion coincides with any of the six points  $O, \Omega, \Omega', A_2, B_2, C_2$ , or their inverses. (Art. 72, Ex. 22.)

5°. If  $O$  is on the circum-circle,  $C' = 0^\circ$ , and the points  $A', B', C'$  are collinear.

6°. Let  $BOC, COA$  and  $AOB$  be equal respectively to  $60^\circ + A, 60^\circ + B, 60^\circ + C$ . Then  $A' = B' = C' = 60^\circ$ ; therefore the *vertices of any triangle may be inverted into those of an equilateral; or one of any given species.*

**117.** In the preceding figure the point  $O$  has been taken inside the triangle. It is easy to verify the analogous angular relations when the centre of inversion is outside  $ABC$ .

It will be observed from the relations of Art. 116 that, if a variable triangle of the species of  $A'B'C'$  be inscribed in the given one, the fixed point in connexion with the figure determined by the method of Art. 19 coincides with the centre of inversion.

**118. Relations between the sides of  $ABC$  and  $A'B'C'$ .**

From similar triangles  $AOB$  and  $A'OB'$ ,

$$AB^2/A'B'^2 = OA \cdot OB/OA' \cdot OB';$$

but  $OA' = r^2/OA$  and  $OB' = r^2/OB$ ,

therefore by substitution

$$AB/A'B' = OA \cdot OB/r^2,$$

or

$$c/c' = OA \cdot OB/r^2.$$

By dividing the similar relations  $a/a' = OB \cdot OC/r^2$  and  $b/b' = OC \cdot OA/r^2$ , we have

$$\frac{a}{b} \bigg/ \frac{a'}{b'} = \frac{OB}{OA} = \text{const.}$$

Hence:—*If the base and ratio of sides of a triangle are given, the base and ratio of sides after inversion are also known.* In each case the locus of the vertex is a circle having the extremities of the base for a pair of inverse points (Art. 70); and since the loci are inverse figures, we have the following important theorem:—

*Every circle and a pair of inverse points invert into a circle and a pair of inverse points; and more generally, A circle and a pair of figures each the inverse of the other with respect to it, retain this relation after inversion from any origin.*

**119. Theorem.**—*Any circle  $X$ , its inverse  $X'$  and the circle of inversion  $O$  are coaxal, i.e. have a pair of common points, real or imaginary.*

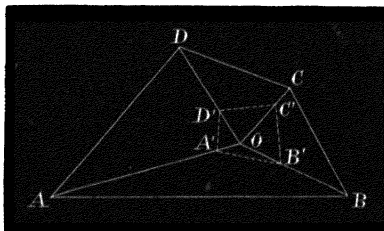
Let  $P$  and  $Q$  be the common pair of inverse points of the circles  $O$  and  $X$ . It is manifest that they are inverse points to  $X'$ . For  $X, P, Q$  invert respectively into  $X', Q, P$ , which by the last Article are a circle and pair of inverse points; therefore, etc,

The theorem requires no proof when the intersections of the circles are real, as the coaxal system is of the common point species.

COR. 1. The circle of antisimilitude is the circle of inversion of either of two given ones with respect to the other; hence, *Two circles and their circles of antisimilitude are coaxal.*

COR. 2. The inverses of the vertices of any triangle with respect to the polar circle, real or imaginary, are the vertices of the pedal triangle; hence, *The circum- and nine-points-circles are inverse figures with respect to the polar circle of the triangle; and the three circles are coaxal.*

120. **Inversion of a System of Four Points.** Let  $A, B, C, D$  and  $A', B', C', D'$  be any four points and their inverses with respect to a given circle of inversion  $O, r$ .



The quadrilaterals  $BCB'C', CDC'D', \dots$  are cyclic. Hence the angular relations:—

$$OA'D' = ODA, OC'D' = ODC,$$

from which we obtain

$$AOC + D + D' = 2\pi \dots\dots\dots(1)$$

Also  $AOC = B + B'$ ; therefore by substituting in (1),

$$B + B' + D + D' = 2\pi; \dots\dots\dots(2)$$

similarly,  $A + A' + C + C' = 2\pi$ ,

or the sums of corresponding pairs of opposite angles of the two quadrilaterals are together equal to four right angles.

The following particular cases are noticed :—

1°. If  $B + D = \pi$ , then also  $B' + D' = \pi$ ; *i.e.*, a cyclic system of points inverts into a cyclic system. Cf. Art. 115.

2°. If  $B' = D'$  and  $A' = C'$  simultaneously,  $A'B'C'D'$  is a parallelogram, and its angles are given by the equations

$$B + D = 2(\pi - B') = 2(\pi - D')$$

and

$$A + C = 2(\pi - A') = 2(\pi - C').$$

NOTE.—The centres of inversion in this case are easily found; for  $AOC = B + B' = B + \pi - \frac{1}{2}(B + D)$ , and  $BOD$  similarly equals  $A + \pi - \frac{1}{2}(B + D)$ ; hence there are two centres of inversion from which the vertices of any quadrilateral invert into the vertices of a parallelogram in an assigned order, viz., the intersections of the known circles  $COA$  and  $BOD$ . Four other points might be similarly found from the intersections of pairs of circles  $BOC$ ,  $AOD$ , and  $AOB$ ,  $COD$ .

3°. A cyclic system of four points may be inverted into the vertices of a rectangle.

121. Relations between the sides of  $ABCD$  and  $A'B'C'D'$ .—By Art. 118,  $BC/B'C' = OB \cdot OC/r^2$  and  $AD/A'D' = OA \cdot OD/r^2$ . Multiplying these relations we have

$$\frac{BC}{B'C'} \cdot \frac{AD}{A'D'} = \frac{OA \cdot OB \cdot OC \cdot OD}{r^4}; \dots\dots\dots(1)$$

similarly,  $\frac{CA \cdot BD}{C'A' \cdot B'D'} = \frac{OA \cdot OB \cdot OC \cdot OD}{r^4}, \dots\dots\dots(2)$

etc. etc.; hence

$$\begin{aligned} BC \cdot AD : CA \cdot BD : AB \cdot CD \\ = B'C' \cdot A'D' : C'A' \cdot B'D' : A'B' \cdot C'D' \dots\dots\dots(3) \end{aligned}$$

COR. 1. If  $A, B, C, D$  be a harmonic system of points on a circle;  $A', B', C', D'$  are also a harmonic cyclic system.

For if the ratios on the left side of (3) are equal, those on the right are also equal.

COR. 2. Combining 3° of the last Article with the previous corollary, it follows that *a harmonic system of cyclic points may be inverted into the vertices of a square.*

#### EXAMPLES.

1. Any two triangles may be placed such that the vertices of the one may be inverses of those of the other taken in any assigned order.

2. Any four points may be inverted into an orthocentric system.  
[For the latter quadrilateral has the following angles:— $A', 90 - A', 180 + A', 90 - A'$ ; hence since  $BOD = A + A', COA = B + 90 - A'$ , and  $A + C + A' + \pi + A' = 180^\circ$ ; the centres of inversion are the intersections of two known circles  $BOD$  and  $COA$ .]

3. Each side of a triangle divided by the perpendicular on it from any origin remains unchanged by inversion.

3a. If the origin is the symmedian point of the one triangle, it is also the symmedian point of the other.

4. If  $\alpha, \beta, \gamma$  denote the perpendiculars from any point on a circle, on the sides of an inscribed triangle, then

$$\beta\gamma \sin A + \gamma\alpha \sin B + \alpha\beta \sin C = 0.$$

[For let  $A'B'C'$  be any three points on a line  $L$ , and  $O$  the origin;  
since 
$$\frac{B'C' + C'A' + A'B'}{OL} = 0,$$

after inversion  $O$  is on the inverse circle  $L'$  and

$$\frac{BC}{a} + \frac{CA}{\beta} + \frac{AB}{\gamma} = 0, \text{ or } \frac{a}{a} + \frac{b}{\beta} + \frac{c}{\gamma} = 0;$$

therefore, etc.]

5. Prove generally for any cyclic polygon that

$$\Sigma(a/a) = 0. \quad (\text{Casey.})$$

6. The inverse of a figure with respect to a line is its reflexion with respect to the line, and is equal in every respect to the given one.

7. The inverses  $A', B', C' \dots$  of the points of intersection  $A, B, C, D$  of any two figures are the corresponding points of intersection of the inverse figures; and the lines  $AA', BB', CC' \dots$  are concurrent at the centre of inversion.

7a. If two curves touch at  $A$ , their inverses touch at  $A'$  the inverse of  $A$ .

8. A circle coincides with its inverse when the circle of inversion is orthogonal to it.

9. A variable chord  $AB$  of a circle, the inverse  $C'$  of a fixed point  $C$  on it and the centre  $O$  are concyclic.

[Since the points  $A, B, C, \infty$  are collinear; their inverses with respect to the given circle are concyclic; i.e.,  $ABC'O$  is a cyclic quadrilateral.]

10. From any point  $P$  on the circum-circle a line is drawn through the symmedian point  $K$ , cutting the sides of the triangle  $ABC$  in  $A', B', C'$ , prove the relation  $\Sigma 1/PA' = 3/PK$ .

[Employ the properties of Ex. 4 and Art. 15, Ex. 1 (3).]

**122. Theorem.** *The inverse of the circum-circle of a triangle  $ABC$  with respect to the in-circle is the nine-points-circle of the triangle  $PQR$  formed by joining the points of contact.*

Let  $X, Y, Z$  be the middle points of the sides of  $PQR$ . From similar triangles we get

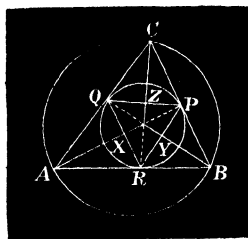
$$OA \cdot OX = OB \cdot OY = OC \cdot OZ = r^2;$$

therefore, etc.



Mr. Piers C. Ward has applied this property in the following elegant proof of *Mannheim's Theorem* :—

Inverting with respect to the in-circle, the circum-circle inverts into  $XYZ$ , that is, a circle passing through



a fixed point  $Z$  and of constant radius ( $=\frac{1}{2}r$ ). It therefore envelopes a circle concentric with  $Z$  whose radius is equal to the diameter of  $XYZ$ ; therefore, etc., by Art. 121, Ex. 7a.

#### EXAMPLES.

1. A variable triangle  $ABC$  is inscribed to one and escribed to another circle; prove that the mean centre of the points of contact  $P, Q, R$  is a fixed point.

[This particular case of *Weill's Theorem* (Art. 53, Ex. 12) is easily seen. For the mean centre of  $P, Q, R$  is the point of trisection of the line joining its circum- and nine-points-centres, both of which are fixed; therefore, etc.]

2. If a quadrilateral  $ABCD$  be inscribed to one circle and circumscribed to another; prove that the mean centre of its points of contact  $P, Q, R, S$  with the inner circle is a fixed point.

[Let  $W, X, Y, Z$  be the middle points of the sides of the cyclic quadrilateral  $P, Q, R, S$ . Then  $W, X, Y, Z$  is a cyclic parallelogram, and is therefore a rectangle. The mean centre of  $P, Q, R, S$  is evidently that of the system  $W, X, Y, Z$ , or the centre of the circle inverse to  $ABCD$  with respect to the other given circle.]

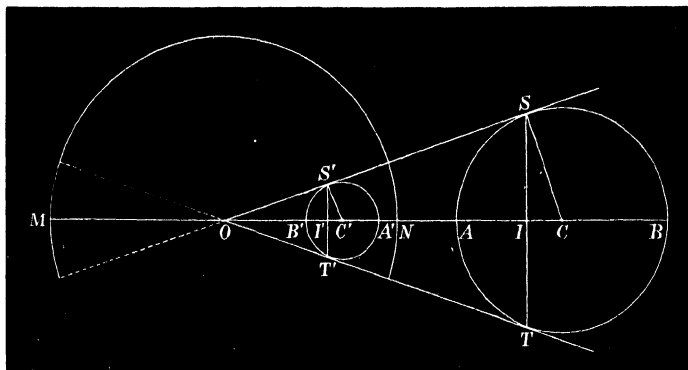
3. The four nine-points-circles of the four triangles formed by taking the vertices of a cyclic quadrilateral in threes pass through a point.

[For the nine-points-circles invert into the circum-circles of the triangles formed by drawing tangents to the circle at the vertices of the quadrilateral; therefore, etc. The more general property for any quadrilateral has been independently demonstrated. Art. 79, Ex. 15.]

## SECTION II.

### ANGLES OF INTERSECTION OF FIGURES AND OF THEIR INVERSES.

123. *The general relations existing between the centres and radii of a circle, its inverse, and the circle of inversion are as follows:—*



Let  $C, C', O$  be the centres of the three circles;  $AB, A'B', MN$  the extremities of their common diameter;  $SS'$  and  $TT'$  the direct common tangents intersecting in  $O$ . Join  $ST$  and  $S'T'$ .

Since  $AB$  and  $A'B'$  are inverse segments with respect to the circle of inversion, the three circles are coaxal. (Art. 114, Ex. 9.)

Let  $I$  and  $I'$  denote the points of intersection of  $ST$  and  $S'T'$  with the line of centres; by comparing equal triangles  $OIS$  and  $OIT'$ , etc., it follows that  $ST$  and  $S'T'$  are both perpendicular to  $AB$ . The quadrilateral  $CSS'I'$  is therefore cyclic; hence the inverse of  $C$  is  $I'$ ; and similarly the inverse of  $C'$  is  $I$  with respect to the circle of inversion, and therefore:—

*The centre  $C$  of any circle inverts into the inverse  $I'$  of the centre of inversion  $O$  with respect to the inverse circle  $C'$ ; and*

*The inverse  $I$  of the centre of inversion  $O$  with respect to any circle  $C$  inverts into the centre  $C'$  of the circle inverse to the circle  $C$ .\**

In the particular case when the inverse circle is a line, the inverse of the centre of a given circle is the reflexion of the origin with respect to the line.

The inverse of  $ST$  is the circle on  $OC'$  as diameter.

Again, by similar triangles  $OC'OC'' = OS/OS' = CS/C'S'$ ,  
or, say  $d/d' = t/t' = r/r' \dots \dots \dots (1)$

To find  $d'$ ,  $t'$ , and  $r'$ , we have

$$d'/d = tt'/t^2 = R^2/(d^2 - r^2),$$

where  $R$  is the radius of inversion.

Hence 
$$d' = \frac{R^2 d}{d^2 - r^2} \dots \dots \dots (2)$$

a relation which gives the position of the centre  $C'$  of the inverse circle.

---

\* Townsend, *Modern Geometry of the Point, Line, and Circle*, 1863, p. 373.

From (1) we have therefore generally

$$\frac{d'}{d} = \frac{r'}{r} = \frac{R^2}{d^2 \sim r'^2} \dots\dots\dots (3)$$

from which the position of the centre and magnitude of the radius of the inverse circle may be determined.

COR. If the centre of inversion is on the circle ;  $d=r$  and  $r'=\infty$ , thus verifying that the inverse of a circle from any origin on its circumference is a right line.

**124. Problem.**—*To invert two circles such that the ratio of the radii of their inverses may be a given quantity  $\kappa$ .*

Let  $r_1, r_2$  be the radii of the given circles ;  $d_1, d_2$  the distances of their centres from the origin  $O$  ;  $R$  the radius of inversion ;  $t_1, t_2$  the tangents, real or imaginary, from  $O$  to the given circles. Then if  $\rho_1, \rho_2$  denote the radii of the inverse circles, we have, by Art. 123,

$$\rho_1 = R^2 r_1 / t_1^2 \text{ and } \rho_2 = R^2 r_2 / t_2^2.$$

Dividing these equations,

$$\frac{\rho_1}{\rho_2} = \frac{r_1}{r_2} \cdot \frac{t_2^2}{t_1^2} = \kappa.$$

The centre of inversion is therefore on a locus such that tangents drawn from any point on it to the given circles have a constant ratio ; *i.e. a circle coaxal with them.*

COR. Any two circles may be inverted into equal circles ; and the locus of the centre of inversion is either circle of antisimilitude.

For when  $\rho_1 = \rho_2$  ;  $t_1^2 / t_2^2 = r_1 / r_2$  ; therefore, etc. (Art. 114, 2°.)

Otherwise thus :—Since a circle and two inverse figures invert into a circle and two inverse figures ; if the origin

be taken on either circle of antisimilitude this circle inverts into a line. Therefore *any two figures the inverse of each other with respect to a circle invert into reflexions of each other with respect to a line.* (Art. 121, Ex. 6.)

## EXAMPLES.

1. Show how to invert any three circles into equal circles.

[The centres of inversion are the points of section of the circles of antisimilitude of the given ones taken in pairs.]

2. How many centres of inversion are there in the solution of Ex. 1?

[The three external circles of antisimilitude are coaxal (Art. 88, Ex. 13), and therefore meet in two real or imaginary points. Also since every two internal and one external circles of antisimilitude are coaxal, there are in all *right* centres of inversion real or imaginary.]

3. Any three circles are unaltered by inversion with respect to their common orthogonal circle. For this reason the latter has been named the *Circle of Self-Inversion* of the given ones.

4. To invert the sides of a triangle into

$\alpha^\circ$ . Three equal circles.

$\beta^\circ$ . Three circles whose radii have any given ratios  $p : q : r$ .

[ $\alpha^\circ$ . The centres of the in- and ex-circles are the four origins.

$\beta^\circ$ . The distances of the origin from the sides are in the inverse ratios  $p : q : r$ .]

**125. Theorem.**—*The tangents at corresponding points  $A$  and  $A'$  of two inverse figures make equal angles with their line of connexion  $AA'$ .*

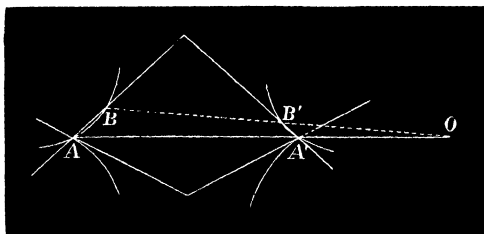
For take the corresponding points  $B$  and  $B'$  on the curves which are consecutive to  $A$  and  $A'$ . Join  $AA'$  and  $BB'$ ; they each pass through  $O$ .

The lines  $AB$  and  $A'B'$  joining consecutive points may be regarded as tangents to the respective curves; also

since  $ABA'B$  is a cyclic quadrilateral and the angle at  $O$  indefinitely small, we have (Euc. III. 22)

$$BAO = OB'A' = AA'B';$$

therefore  $TAA'$  is an isosceles triangle.



**126. Theorem.**—*The angle of intersection of two curves is similar\* to that of their inverses at the corresponding point.*

For the angle between any two curves is the angle between the tangents at their points of intersection.

But the tangents determine two isosceles triangles (Art. 125) on the line  $AA'$ ; therefore, etc.

If the centre of inversion is external or internal to both circles the angle remains unaltered; if on the other hand it is external to either and internal to the other, the angles of intersection before and after inversion are supplemental.

\* "The angle of intersection of two circles undergoes as a *figure* no change of form under the process of inversion, but often does as a *magnitude*, change into its supplement, under that process.

"In the application of the theory of inversion to the geometry of the circle, this circumstance must always be attended to."

"The two cases of contact, external and internal, come of course under it as particular cases; and in but one case alone, that of *orthogonal* intersection, which presents no ambiguity, can the precaution ever be entirely dispensed with." Townsend's *Modern Geometry of the Point, Line, and Circle*, Art. 407.

127. Amongst the various results which follow from the preceding Articles, we note

- 1°. Any two circles meeting at an angle  $\alpha$  invert from either point of intersection into two lines inclined at the same angle, *e.g.* two orthogonal circles into two lines at right angles.
- 2°. Three mutually orthogonal circles, *e.g.* the three *real* polar circles of the triangles formed from an orthocentric system of points, invert from any of their points of intersection into a circle and two perpendicular diameters.
- 3°. Any three circles invert from any centre on their common orthogonal circle into three others whose centres are collinear; the line of collinearity being the inverse of the common orthogonal circle.
- 4°. A system of circles having more than one orthogonal circle inverts into a system having more than one orthogonal line.
- 5°. In 4° the intersections of the common orthogonal circles are evidently the limiting points of the given system which is coaxal. (Art. 86.)

Hence for any centre of inversion:—

- a°. *A coaxal system inverts into a coaxal system; or*
- b°. *A circle and a pair of inverse points invert into a circle and a pair of inverse points;*

and for a centre of inversion at either of the limiting points:—

- c°. *A coaxal system inverts into a concentric system, the common centre being the inverse of the second limiting point with respect to the circle of inversion.*

- 6°. *A system of concurrent lines inverts into a coaxal system of the common point species, the common points being the centre of inversion and the inverse of the point of concurrence.*
- 7°. *An angle and its bisectors invert into two circles and their circles of antisimilitude. (Art. 109.)*
- 8°. *If two circles, concentric with the extremities of the third diagonal of a cyclic quadrilateral, are described cutting the given one orthogonally; they are mutually orthogonal, and their points of intersection  $O_1$  and  $O_2$  are therefore inverse points with respect to the given circle. Hence if we take  $O_1$  and  $O_2$  as centres of inversion we arrive at the following results:—The three circles invert into a circle and two rectangular diameters; the vertices of the quadrilateral, which are inverse points with respect to the circles, invert into inverse points in the same order with respect to the lines, i.e. form the vertices of a rectangle. Thus the vertices of any cyclic quadrilateral may be inverted into those of a rectangle, and the centres of inversion are inverse points with respect to the circle.*
- 9°. *A circle may invert into a circle having its centre at a given point  $A$ .*

For let  $A'$  the inverse of  $A$  be the centre, and  $AA'$  the radius of inversion. Then the given circle and pair of points  $A$  and  $A'$  inverse to it, invert into a circle and a pair of inverse points; but the inverse of the centre of inversion  $A'$  is at infinity; therefore  $A$  is the centre of the inverse circle.



- 10°. Two parallel lines invert into two circles touching externally if the origin is between the lines; and internally if the lines are on the same side of the origin.
- 11°. If a quadrilateral  $ABCD$  inverts into a parallelogram from an origin  $O$ ; the pairs of circles  $BOC$ ,  $AOD$  and  $COA$ ,  $BOD$  touch at  $O$ .\*

## SECTION III.

## ANHARMONIC RATIOS UNALTERED BY INVERSION.

128. **Theorem.**—If  $A, B, C, D$  be any four concyclic points and  $A', B', C', D'$  their inverses with respect to any circle of inversion, then

$$BC \cdot AD : CA \cdot BD : AB \cdot CD = B'C' \cdot A'D' : C'A' \cdot B'D' : A'B' \cdot C'D'.$$

This property has been shown to hold for *any* four points and their inverses, and is therefore true in the particular case when they lie on a circle; hence the anharmonic ratios of four concyclic points are equal to the anharmonic ratios of their inverses with respect to any circle of inversion. Particular cases have been noticed in Art. 121, Cors. 1, 2.

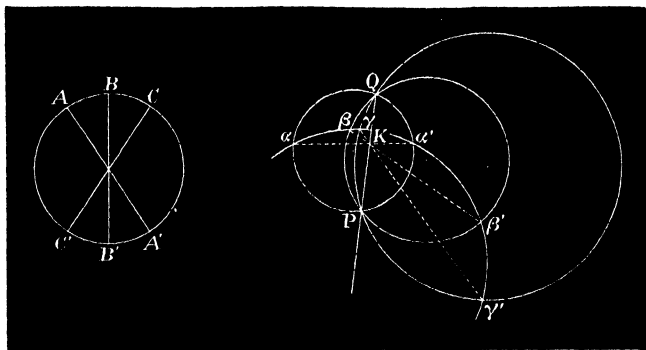
129. **Problem.**—To invert a regular cyclic polygon  $ABC... from any origin  $P$ .$

The circumcircle  $ABC...$  inverts into a circle  $\alpha\beta\gamma...$ ; the diameters  $AA', BB', CC'...$  into circles passing through the origin  $P$  and cutting  $\alpha\beta\gamma...$  orthogonally in  $\alpha\alpha', \beta\beta', \gamma\gamma'...$

---

\* Hence a construction for the required centres of inversion.

They therefore pass through  $Q$  the inverse of  $P$  with respect to the inverse circle and thus form a coaxal system of the common point species. (Art. 127, 6°.) Also the chords  $aa'$ ,  $\beta\beta'$ ,  $\gamma\gamma'$ ... meet in a point  $K$  on  $PQ$  (Art. 72, Ex. 6).



On the primitive figure any side  $BC$  of the polygon and any diameter  $AA'$  meet the circle in a harmonic row of points; therefore (Art. 128) on the inverse figure  $\beta\gamma aa'$  is an harmonic row; hence  $\beta a/\gamma a = \beta a'/\gamma a'$ , or, by Euc. III. 22, the diagonal  $aa'$  of the quadrilateral is the locus of a point such that its distances from either pairs of sides which meet at its extremities are proportional to the lengths of the sides; similarly for the quadrilaterals  $\gamma\delta\beta\beta'$ , etc. Therefore the distances of the point  $K$  from the sides of the polygon  $a\beta\gamma$ ... are proportional to the sides.

For an *Harmonic Quadrilateral*,  $K$  is evidently at the intersection of the diagonals; and the inverse of the regular polygon possessing, as has been shown, a corre-

sponding and more general property has been termed by Casey an *Harmonic Polygon*.

**Definitions.**—The point  $K$  is called the *Symmedian Point of the Polygon*; and if the ratio of any perpendicular from  $K$  to half the side on which it falls is  $\tan \omega$ , then  $\omega$  is the *Brocard Angle of the Polygon*.

For the properties of harmonic polygons the reader is referred to Casey's *Sequel to Euclid*, Supplementary Chapter, Section VI.

**130. Cosymmedian Triangles.**—Let  $ABC$  be a triangle  $K$ , its symmedian point, and let the lines  $AK, BK, CK$  meet the circum-circle again in  $A', B', C'$ . If the circle of inversion be  $K, \rho$  where

$$KA \cdot KA' = KB \cdot KB' = KC \cdot KC' = -\rho^2,$$

the vertices of  $ABC$  invert into  $A', B', C'$ .

Also since  $BCAA'$  is a harmonic quadrilateral, therefore  $B'C'A'A$  is harmonic, or  $A'A$  is a symmedian of the triangle  $A'B'C'$ ; similarly the other symmedians are  $B'B$  and  $C'C$ .

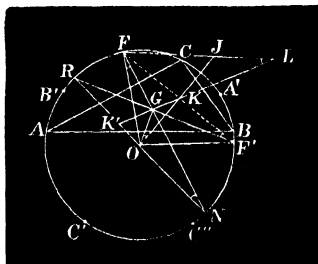
It appears thus that the two triangles have the same symmedian lines, symmedian point, Brocard Circle, Brocard Angle, Brocard Points, etc. On account of these relations they have been termed *Cosymmedian Triangles*.\*

---

\* Their properties were first stated by Casey before the Royal Irish Academy in December, 1885. A further account of them will be found in Milne's *Companion*.

## EXAMPLE.

1. If  $ABC$  be a triangle and  $G$  its centroid;  $AA'$ ,  $BB'$ ,  $CC'$  chords of the circum-circle passing through  $G$ ; the symmedian point of  $A'B'C'$  is on the diameter which contains Tarry's point. (Vigarié.)



[Let the circle be self-inverted from  $G$  as origin and the points  $A$ ,  $B$ ,  $C$  invert into  $A'$ ,  $B'$ ,  $C'$  respectively. Let  $AA'$ ,  $BB'$ ,  $CC'$  be the symmedian chords meeting in  $K$ .

If a circle  $CGC''$  meet  $GK$  in the point  $L$  then

$$KG \cdot KL = KC \cdot KC'';$$

and similar relations hold for the circles  $AGA''$  and  $BGB''$ ; therefore these three circles meet in a second common point  $L$ , which is the inverse of  $K'$ , the symmedian point of  $A'B'C'$ .

Let  $J$  be the inverse of  $K$  with respect to the circum-circle  $ABC$ , and it follows that  $KO \cdot KJ = KG \cdot KL =$  the power of  $K$  with respect to the circum-circle. Hence  $OGJL$  is a cyclic figure, and the angle  $GOK = L$ .

It has been shown (Art. 67, Ex. 18) that Tarry's point on the circum-circle corresponds to  $O$  the circum-centre on the Brocard Circle with respect to  $ABC$  and Brocard's first triangle, and that  $G$  is their common centroid; hence angle  $GNO = GOK$  and  $GRO = GKO = GF'O$ . Therefore  $OGKF'$  is a cyclic quadrilateral, and (Euc. III. 21) the points  $F$ ,  $K$ ,  $F'$  are collinear. Therefore  $KO \cdot KJ = KG \cdot KL = KF \cdot KF'$  or  $F$ ,  $J$ ,  $L$  are collinear, the line being the inverse of the circle  $OGKF'$  with respect to  $K$  as origin.

Now the circum-circles of  $ABC$  and  $GFL$  cut each other orthogonally since the angle  $OFG = L$ ; hence the inverse of the latter from  $G$  is the diameter  $NR$ , and therefore  $L$  inverts into a point  $K'$  on it; therefore, etc.

This solution is due to M'Cay.\*]

#### MISCELLANEOUS EXAMPLES.

1. The six circles that can be described to touch three given ones  $A, B, C$ , two externally and one internally and two internally and one externally, are in pairs the inverses of one another with respect to the common orthogonal circle of  $A, B, C$ .

[Invert with respect to the common orthogonal circle of  $A, B, C$ , and since  $A, B, C$  remain unaltered after inversion, three of the circles of contact invert into the remaining three; therefore, etc.]

2. The eight circles of contact with  $A, B, C$  have a common circle of antisimilitude.

[As in Ex. 1 they are in pairs the inverses of each other with respect to the common orthogonal circle of  $A, B$ , and  $C$ .]

3. Three circles are described touching the ex-circles of a triangle, two externally and one internally; prove that they each pass through the centre of Taylor's Circle.

[Invert with respect to Taylor's Circle and the circles in question invert into the remaining circles of contact, which in this case are the sides of the triangle; and since the circles invert into lines they each pass through the centre of inversion.]

4. If  $ABC$  be a triangle;  $C, \rho$  a circle of inversion,  $A'$  and  $B'$  the inverses of  $A$  and  $B$ ; to prove that

$$2s = \rho^2 \sin C/r'$$

where  $r'$  is the radius of the in-circle of  $A'B'C$ .

[We have  $AC = \rho^2/A'C$ ,  $BC = \rho^2/B'C$  and  $AB/A'B' = \rho^2/A'C \cdot B'C$ , hence by addition

$$2s = \rho^2 \cdot \frac{A'C + B'C + A'B'}{A'C \cdot B'C} = \rho^2 \sin C/r'.]$$

---

\* "Mathematical Questions with their Solutions," from the *Educational Times*, vol. lii., p. 73.

5. **Mannheim's Theorem.\*** Having given the vertical angle  $C$  and radius  $r'$  of the in-circle of a triangle  $A'B'C$ ; the envelope of the circum-circle is a fixed circle.

[From Ex. 4 by inverting from the vertex with respect to a circle of inversion  $C, \rho$ , the inverse of the circum-circle is the base  $AB$  of a triangle of known perimeter; and since the inverse envelopes a circle, viz., the ex-circle of the triangle  $ABC$ ; therefore, etc.]

6. A variable circle touches the base of an isosceles triangle at its middle point; prove that the chords of intersection with the sides that meet within the circle envelope a fixed circle. (M'Vicker.)

[See the property of Art. 61, Ex. 1.]

6a. By inverting from the vertex derive *Mannheim's Theorem*.

7. Two circles meet at an angle  $\omega$ , and are such that  $2 \cos \omega = \sqrt{r/R}$ ; prove that a triangle may be inscribed to one and circumscribed to the other. Hence find the locus of a point from which two circles may be inverted into two others, so that a triangle may be inscribed to one and circumscribed to the other.

8. A variable chord  $XX'$  of a circle  $O, r$  passes through a fixed point  $Q$ ; to prove that the circum-circles of the triangles  $QOX$  and  $QOX'$  envelope coaxial systems.

[Let  $P$  be the inverse of  $Q$  with respect to the given circle. The circles in question invert into the right lines  $PX$  and  $PX'$ , which by Art. 72. Cor. 5, touch each of two concentric systems, viz., the in- and ex-circles of the triangle  $PXX'$ .]

9. Prove that the vertices of a triangle and the reflexions  $O_1, O_2, O_3$  of any point  $O$  with respect to the sides may be inverted into the vertices of a triangle and three collinear points on the sides. (Russell.)

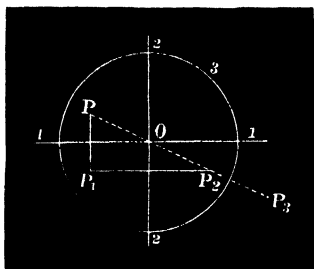
[The circle  $BCO_1, CAO_2, ABO_3$  meet in a point  $P$  (Art. 79, Ex. 15), which is seen from Euc. III. 22 to be on the circum-circle of  $O_1O_2O_3$ . Inverting from  $P$ ; therefore, etc.]

\* This well-known property is thus seen to be the inverse of:--*Having given the vertical angle  $C$  and either of the quantities  $s$  or  $s - c$ ; the envelope of the base is a circle.*

10. Any triangle  $ABC$  and a Simson line  $XYZ$  may be inverted from the *pole of the line* into a triangle  $X'Y'Z'$  and Simson line  $A'B'C'$ .

11. If four circles be mutually orthogonal, and if any figure be inverted with respect to each in succession; the fourth inversion will coincide with the original figure.

[The following proof has been given by M'Cay:—Invert the four orthogonal circles from a point of intersection of any two of them. The latter invert into rectangular lines; a third circle becomes one  $\rho$ , cutting these lines at right angles; and the fourth after inversion ( $\rho'$ ), since it cuts the third at right angles and is concentric with it, satisfies the relation  $\rho^2 + \rho'^2 = 0$  or  $\rho'^2 = -\rho^2$ .



Let  $P_1, P_2, P_3$  denote the successive inversions of the point  $P$  on the inverse figure; since  $OP_2 \cdot OP_3 = \rho^2$  and  $OP_2 = -OP$ , therefore  $OP \cdot OP_3 = -\rho^2$ , or the inverse of  $P_3$  with respect to the imaginary circle of radius  $i\rho$ , whose centre is at  $O$ , coincides with  $P$ ; therefore, etc.]

12. "The centres of the four circles circumscribed about the four triangles formed by four right lines are concyclic." Prove this theorem by inversion from the point  $P$  common to the four circum-circles, and show that the circle passes through  $P$ .

[It is evident that, 1°, the four lines invert into four circles passing through  $P$ ; 2°, the four circles into lines joining the remaining pairs of intersections of the circles in 1°; 3°, the centres of the four circles into the reflexions of  $P$  with respect to the four

lines on the inverse figure by Art. 123 ; but these are collinear ; therefore, etc.]

13. Let  $T$  be a common tangent to two circles,  $t$  and  $t'$  the tangents to them from any point  $O$  ; if the circles are inverted from  $O$  as origin prove that  $T^2/tt'$  is unaltered.

14. The vertex  $C$  of a given angle  $ACB$  is fixed ; required to find the envelope of the circle  $ACB$  where  $A$  and  $B$  are points on a given line.

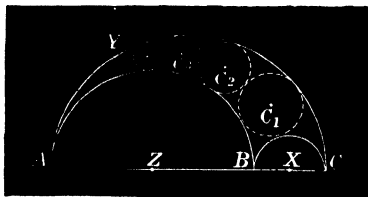
15. A chord  $AB$  of a circle passes through a fixed point  $P$  ; find the locus of the point of intersection of the circles passing through  $P$  and touching the given one at  $A$  and  $B$ .

16. If two circles be inverted into any two others ; for each pair the square of the common tangent divided by the product of the diameters are equal.

[Compare Art. 126 and Art. 4, footnote.]

17. Prove Casey's relation among the common tangents to four circles all of which are touched by a fifth (Art. 7) by the inversion of a system of four circles touching a line.

18. Draw two parallel lines and describe a number of circles touching the lines and each other in succession. Invert this system from a point on a diameter of any circle perpendicular to the lines and deduce the following theorem :—



$A, B, C$  are three collinear points, and circles  $X, Y, Z$  are described on the segments  $BC, CA, AB$  respectively. A system of circles is drawn as in figure to touch each other and the given ones, if  $C_n, \rho$  denote the  $n$ th circle to prove that the distance of its centre from  $AB = 2n\rho$ . (Pappus.)



19. If three circles  $Ar_1$ ,  $Br_2$ ,  $Cr_3$  touch one another in pairs ; prove by inversion that the radii of the circles which touch them with contacts of similar species are

$$\frac{r_1 r_2 r_3}{\Sigma r_1 r_2 \pm 2\Delta}$$

where  $2\Delta$  is the area of the triangle  $ABC$ .

[Invert from the point of contact of  $Br_2$ ,  $Cr_3$  with a radius equal to the tangent to  $Ar_1$  ; etc.]

20. The rectangle under the distances of the ex-centre of similitude of two circles from their radical axis and in-centre of similitude is equal to the constant product of antisimilitude.

[The circle of similitude inverts from either centre of similitude into the radical axis of the given circles.

20a. Prove that the poles of the radical axis of two circles with respect to the circles are harmonic conjugates with respect to the centres of similitude.

[This is the inverse of the theorem :—*The polars of either centre of similitude with respect to two circles are equidistant from their radical axis ; the circle of antisimilitude being taken as circle of inversion.*]

21. A variable circle  $ABCD$  touching two fixed circles externally meets their radical axis in  $L$  and  $O$  and the pair of transverse common tangents in  $A$ ,  $C$  and  $B$ ,  $D$  respectively ; prove the following properties of the figure :—

1°. The limiting points  $M$  and  $N$  of the circles are the middle points of the parallel sides of the quadrilateral  $PQRS$ .

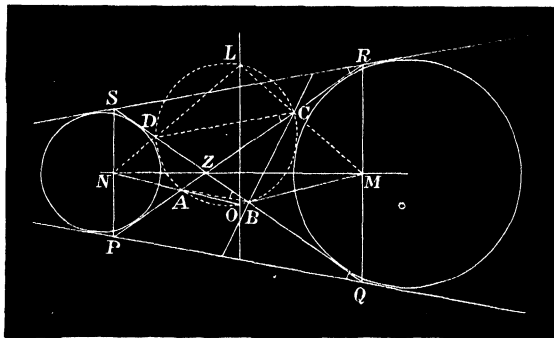
2°. The lines  $AB$  and  $CD$  move parallel to the direct common tangents  $PQ$  and  $RS$  respectively.

3°. The vertices of  $ABCD$  lie on the lines joining  $O$  and  $L$  to the limiting points.

4°.  $BC$  and  $AD$  envelope circles concentric with  $M$  and  $N$  respectively.

To prove 1°. Since the four common tangents to the two given circles form a common escribed quadrilateral, the diagonals of which are concurrent with the diagonals of the corresponding inscribed quadrilaterals ; therefore, etc. See Art. 67, Cor. 6.

2°. Let the points  $A$  and  $B$  and the given circles be numbered 1, 2, 3, 4. Apply Casey's relation connecting the common tangents to four circles all touched by a fifth and reduce, it follows that  $AZ + BZ \propto AB$ . Hence  $AB$  is constant in direction and  $PQ$  is a particular position of it, therefore  $AB$  and  $PQ$  are parallel; similarly  $CD$  and  $RS$  are parallel.



3°. To prove that the points  $D, L, N$  are collinear. Invert the figure from  $D$  as origin. The circles, their radical axis and pair of inverse points invert into three coaxial circles, one of which passes through the origin, and their limiting points; also the circle  $ABCD$  inverts into the direct common tangent of the latter system. It follows easily (Art. 92, Ex. 5) that the inverses of  $N$  and  $L$  pass through  $D$ : therefore, etc.

4°.  $BM$  bisects externally the base angle  $B$  of the triangle  $ZBC$ , since  $LO$  bisects internally the vertical angle of the isosceles triangle  $LMN$ ; similarly  $CM$  bisects externally the other base angle, therefore  $M$  is the ex-centre of  $BCZ$ .

NOTE.—This property, communicated by Mr. Charles M'Vicker, is a manifest extension of Mannheim's Theorem. For if either of the circles is reduced to a point  $Z$ , we have of the triangle  $BCZ$  the vertical angle  $Z$  fixed in magnitude and position and the ex-circle; since the variable circum-circle  $BCZ$  (*i.e.*  $ABCD$ ) envelopes a circle to which the vertex and centre of the ex-circle are a pair of inverse points; therefore, etc.

22. Prove the converse of *Casey's Theorem* (Art. 7), showing the relation which holds between the common tangents to four circles, all of which are touched by a fifth.

[Invert the circles 1, 2, 3 into equal circles (Art. 124)  $A, r; B, r; C, r$ ; and find the inverse  $D, r_1$  of 4 with respect to the same circle of inversion. The relation  $\Sigma \bar{2}\bar{3} \cdot \bar{1}4 = 0$  holds for the four circles after inversion (Art. 126); also the tangents  $\bar{2}\bar{3}, \bar{3}\bar{1}, \bar{1}\bar{2}$  are equal to the sides of the triangle  $ABC$  formed by joining the centres of the equal circles. Now describe a circle concentric with  $D$  and a radius equal to  $r \sim r_1$ , and the tangents from  $A, B, C$  to it are respectively equal to  $\bar{1}4, \bar{2}4, \bar{3}4$ . Hence the general relation has been reduced to the corresponding one for three points and a circle. It is easy to see that the circum-circle of  $ABC$  touches  $D, r \sim r_1$ ; for by the converse of Ptolemy's Theorem the limiting points of the two circles are on  $ABC$ ; therefore, etc. Fry.]

NOTE.—The method of inversion so useful in Modern Geometry was discovered by the Rev. Dr. Stubbs of Trinity College, Dublin, in the year 1843. His valuable memoir on the subject is to be found in the *Philosophical Magazine*, Nov., 1843, p. 338. About the same time, Dr. Ingram published his researches in the *Transactions of the Dublin Philosophical Society*. See vol. i., p. 145.

## CHAPTER XII.

### GENERAL THEORY OF ANHARMONIC SECTION.

#### SECTION I.

##### ANHARMONIC SECTION.

**131. Definitions.**—Let a line  $AB$  be divided by two variable points  $C$  and  $D$  such that  $AC/BC \div AD/BD$  is a constant ratio ( $=\kappa$ ). The value of  $\kappa$  is thus

$$-CA \cdot BD/BC \cdot AD,$$

and is termed the *Anharmonic Ratio* in which the segment  $AB$  is divided by the points  $C$  and  $D$ . Similarly the anharmonic ratio of  $CD$  divided at  $A$  and  $B$  is

$$CA/DA \div CB/DB \text{ or } -CA \cdot BD/BC \cdot AD.$$

The points  $C$  and  $D$  are *Conjugate* or *Corresponding Points* in the Row  $A, B, C, D$ , and  $AB$  and  $CD$  are *Conjugate Segments*. It is obvious that conjugate segments divide each other *Equianharmonically*, i.e. the anharmonic ratio of  $AB$  divided at  $C$  and  $D$  is equal to that of  $CD$  divided at  $A$  and  $B$ .

**132.** Let the four points  $A, B, C, D$  be divided into three pairs of opposite segments  $BC, AD$ ;  $CA, BD$ ;  $AB, CD$ ; then the anharmonic ratios of

$BC$  divided in  $A$  and  $D = BA/CA \div BD/CD = \lambda$ , (1)

$CA$  divided in  $B$  and  $D = CB/AB \div CD/AD = \mu$ , (2)

and  $AB$  divided in  $C$  and  $D = AC/BC \div AD/BD = \nu$ , (3)

or their reciprocals; since a segment divided in  $A$  and  $D$  is divided in the reciprocal anharmonic ratio by  $D$  and  $A$ .

These three fractions  $\lambda$ ,  $\mu$ ,  $\nu$  and their reciprocals are the six anharmonic ratios of the four points  $A$ ,  $B$ ,  $C$ ,  $D$ .

NOTE.—Let a line  $AB$  be divided internally in a variable point  $X$  and externally in  $X'$  such that  $AX/BX = k \cdot AX'/BX'$ . As  $X$  approaches  $B$ ,  $AX/BX$  increases; therefore the conjugate point  $X'$  approaches  $B$  simultaneously. For let  $AX' = a$  and  $BX' = b$  and we have

$$\frac{a-x}{b-x} > \text{ or } < \frac{a}{b} \text{ according as } a > \text{ or } < b.$$

but  $a > b$ , thus it follows that as  $X'$  moves towards  $B$  the ratio  $AX'/BX'$  continually increases, and becomes infinitely great when the variable point coincides with  $B$ . Here also it coincides with its conjugate  $X$ , and the point  $B$  is thus a *Double Point* of the systems described by the variables  $X$  and  $X'$ . Similarly  $A$  is a double point.

Again, as  $X'$  recedes from  $B$  on the line produced,  $X$  approaches  $M$  the middle point of  $AB$ . In the limit when  $X'$  is at infinity and  $AX'/BX'$  therefore equal to unity, its conjugate  $X (= P)$  divides the line in the simple ratio  $AP/PB = k$ . Similarly when  $X$  moves to infinity, its conjugate  $X' (= Q)$  gives the relation  $AQ/BQ = 1/k$ ; and the two points whose conjugates are at infinity are isotomic conjugates with respect to  $AB$ .

We may note here, and we shall see presently, that when the corresponding points of the two systems move in the same direction the *double points are imaginary*.

**133. Problem.**—To express all the Anharmonic Ratios of  $ABCD$  in terms of any one of them ( $\lambda$ ).

Since  $BC \cdot AD + CA \cdot BD + AB \cdot CD = 0$ ;  
dividing by  $AB \cdot CD$ , we have

$$\frac{BC \cdot AD}{AB \cdot CD} + \frac{CA \cdot BD}{AB \cdot CD} + 1 = 0,$$

whence on substituting from Art. 132

$$-\mu - 1/\lambda + 1 = 0.$$

Thus generally it follows, by dividing the above equation by each of its terms, that

$$\mu + 1/\lambda = 1; \quad \nu + 1/\mu = 1; \quad \lambda + 1/\nu = 1.$$

The six ratios are therefore

$$\lambda, 1/\lambda, (\lambda - 1)/\lambda, \lambda/(\lambda - 1), 1 - \lambda, 1/(1 - \lambda).$$

These may be expressed as trigonometrical functions of an angle. For let  $\lambda = \sec^2 \theta$ . Then the ratios taken in the above order reduce to the following:—

$$\sec^2 \theta, \cos^2 \theta, \sin^2 \theta, \operatorname{cosec}^2 \theta, -\tan^2 \theta, -\cot^2 \theta.$$

If two of the ratios are equal, e.g.  $\lambda = (\lambda - 1)/\lambda$ , then  $\lambda^2 - \lambda + 1 = 0$  and  $\lambda = \omega$  or  $\omega^2$ , the imaginary cube roots of unity. In this case the three pairs of ratios have the values  $\omega$  and  $\omega^2$ .

If  $\lambda = -1$  the points form an harmonic row, and the remaining ratios are  $-1, -2, -1/2, 2, 1/2$ .

In speaking of *the* anharmonic ratio of four points on a line the order in which the points are taken is to be understood. Dr. Salmon introduced the convenient notation  $[ABCD]$  to denote the ratio into which  $AB$  is divided by  $C$  and  $D$ .  $[ABCD]$  is equivalent to  $AC/BC \div AD/BD$ , and  $[ABCD] \cdot [ABDC] = 1$ .

#### EXAMPLES.

1. To prove that  $[ABCD] = [BADC] = [DCBA] = [CDAB]$ ; and hence when any two constituents of four points are interchanged, the anharmonic ratio of the system remains unaltered, provided the remaining pair be likewise interchanged.

2. If  $[ABCD] = [ABDC] = \kappa$ ; find the value of  $\kappa$ .

[It is plain that  $\kappa$  is equal to its reciprocal, and is therefore unity. The four points form in this case an harmonic system.]

3. To prove for any collinear system of points  $A, B, C, D, E \dots$  that  $[ABCE]/[ABCD]=[ABDE]$ .

[Expanding the ratios on the left side and reducing ; therefore, etc.]

4. For any two collinear systems of points  $A, B, C, D, E \dots$   $A', B', C', D', E' \dots$  having given  $[ABCD]=[A'B'C'D']$  and  $[ABCE]=[A'B'C'E']$ , to prove that

$[BCDE]=[B'C'D'E']$ ,  $[CADE]=[C'A'D'E']$ ,  $[ABDE]=[A'B'D'E']$ .

[By Ex. 3.]

5. If  $[ABCD]=[ABC'D']$ , prove that  $[ABCC']=[ABDD']$ .

[Expanding the ratios the required result follows by alternation.]

6. If in Ex. 4  $[ABCD]=[A'B'C'D']$ ,  $[ABCE]=[A'B'C'E']$ ,  $[ABCF]=[A'B'C'F']$ , etc., etc. ; prove that

$[ADEF]=[A'D'E'F']$ ,  $[BDEF]=[B'D'E'F']$ , etc. ....(1)  
and  $[DEFG \dots]=[D'E'F'G' \dots]$ .

7. If a segment  $MN$  is divided equianharmonically by pairs of points  $A, A', B, B', C, C'$ , etc. ; to prove that

1°.  $[MABC \dots]=[MA'B'C' \dots]$  and  $[NABC \dots]=[NA'B'C']$ .

2°.  $[ABCD \dots]=[A'B'C'D' \dots]$ .

[Since  $[MNAA']=[MNBB']=[MNCC']=\dots$  etc., by Ex. 5.  $[MNAB]=[MNA'B']$ ;  $[MNAC]=[MNA'C']$ , etc. Hence by division we have  $[MABC]=[MA'B'C']$ , etc. ...]

To prove 2°. We have by 1°  $[MABC]=[MA'B'C']$  and  $[MABD]=[MA'B'D']$ , therefore by division  $[ABCD]=[A'B'C'D']$ .

8. If a segment  $MN$  is divided harmonically by points  $A$  and  $A'$ ,  $B$  and  $B'$ ,  $C$  and  $C'$  ; to prove that the anharmonic ratio of four of the six points taken in any order is equal to that of their four conjugates,  $[ABCC']=[A'B'C'C']$ .

[By Ex. 7.  $[MABC]=[MA'B'C']$  ; but (hyp.)  $C$  and  $C'$  are interchangeable, therefore  $[MABC]=[MA'B'C']$  ; dividing these equations, therefore, etc., as in Ex. 4.]

9. To prove the converse of Ex. 8, i.e., for any six collinear points  $A, B, C, A', B', C'$ , if the anharmonic ratio of any four is equal to that of their four conjugates  $[CABA']=[C'A'B'A']$  then

1°. The anharmonic ratio of every four is equal to that of their four conjugates.

2°. The segments  $AA'$ ,  $BB'$ ,  $CC'$  have a common segment of harmonic section.

[To prove 1°. By hyp. since  $[CABA'] = [C'A'B'A]$ ; on rearranging, by Ex. 1, we get  $[AA'BC] = [A'AB'C'] = [AA'C'B']$ . Therefore by alternation (Ex. 5)  $[AA'BC'] = [AA'CB'] = [A'AC'B]$ ; similarly for all other combinations. To prove 2°. Let  $MN$  divide the segments  $AA'$  and  $BB'$  harmonically, it divides  $CC'$  also harmonically. For  $[MABA'] = [MA'B'A]$  (by Ex. 7) and  $[NABA'] = [NA'B'A]$ ; also by 1°  $[CABA'] = [C'A'B'A]$  and  $[C'ABA'] = [C'A'B'A]$ , hence (Ex. 6)  $[MNCC'] = [MNC'C]$ ; therefore, etc. (Ex. 2)].

10. Show generally for two equianharmonic systems if any two conjugates  $A$  and  $A'$  are interchangeable, *e.g.*, if  $[ABCD] = [A'B'C'D]$  and  $[A'BCD] = [AB'C'D]$  that

1°. Every four are equianharmonic with their four opposites;

2°. The segments  $AA'$ ,  $BB'$ ,  $CC'$ ,  $DD'$  have a common segment of harmonic section.

[By the method of Ex. 9.]

## SECTION II.

### ANHARMONIC SECTION OF AN ANGLE.

134. It has been explained in Art. 3 that the anharmonic ratio of four points  $A, B, C, D$  is equal to that of the pencil  $O.ABCD$  formed by joining them to any point  $O$ . It follows then that all the properties of four collinear points stated in the previous section involve correlative properties of a pencil of rays, and that the latter are immediately derived from the former by aid of the equation

$$BC.AD : CA.BD : AB.CD$$

$$= \sin \widehat{BC} . \sin \widehat{AD} : \sin \widehat{CA} . \sin \widehat{BD} : \sin \widehat{AB} . \sin \widehat{CD}.$$

Also by describing a circle through the vertex  $O$  of the

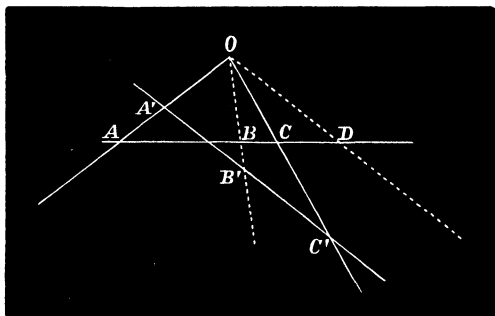


pencil  $O.ABCD$ , and denoting by  $A, B, C, D$  the points where it meets the legs of the pencil again; since the sines of the angles at  $O$  are in the ratios of the chords opposite to them we may further obtain from the anharmonic properties of collinear points corresponding relations amongst points which lie on a circle.

135. The following properties will appear evident:—

1°. All transversals to a pencil of rays are cut equianharmonically.

2°. A transversal to a pencil drawn parallel to one of its rays  $D$  is divided by the remaining three in the simple ratio  $AC/BC$ ; which is the anharmonic ratio of the pencil.



3°. In 2°, if the pencil is harmonic, any transversal  $A'B'C'$  parallel to  $D$  is such that  $A'B' = B'C'$ .

4°. For any two equianharmonic rows of points  $A, B, C, D, \dots$  and  $A', B', C', D', \dots$ , if the lines  $AA', BB',$  and  $CC'$  are concurrent at  $O$ ;  $DD'$  and all other lines joining corresponding points of the given systems pass through  $O$ .

[This important property is the converse of 1° and follows easily by an indirect proof.]

**136. Theorem.**—*If two lines be divided equianharmonically such that a pair of corresponding points coincide at their intersection  $[OABC\dots]=[OA'B'C'\dots]$  the systems are in perspective; and reciprocally if two equianharmonic pencils are such that a pair of corresponding rays coincide on the lines joining their vertices they are in perspective.*

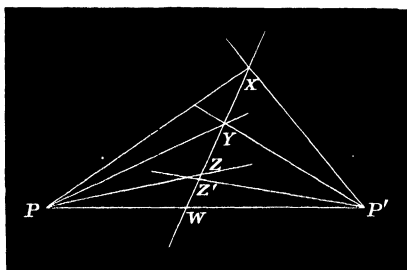
Let  $AA'$  and  $BB'$  meet in  $P$ . Join  $PC$ , and if possible let  $PC$  cut the other axis in  $C''$ . Then

$$[OABC]=[OA'B'C''],$$

since the rows are in perspective. But

$$[OABC]=[OA'B'C'] \text{ (hyp.)};$$

therefore  $[OA'B'C']=[OA'B'C'']$ , i.e.  $C'$  and  $C''$  coincide. Reciprocally for any two pencils  $P.ABC, \dots$  and  $P'.A'B'C', \dots$  if the rays  $A, A'$  and  $B, B'$  intersect respec-



tively in  $X$  and  $Y$ , it follows that  $C$  and  $C'$  meet on the line  $XY$ .

Otherwise thus :—The rows  $[XYZW]$  and  $[XYZ'W]$  are equianharmonic; therefore  $Z$  and  $Z'$  coincide.

**COR. 1.** If two pencils are equianharmonic, any two rows passing through the intersection of a pair of corresponding rays are in perspective.

COR. 2. Through a given point  $P$  a line may be drawn across a triangle  $ABC$ , cutting its sides in the points  $Q, R, S$ , such that  $[PQRS] = \text{a given anharmonic ratio}$ .

[For the pencil  $(A.PQRS)$  formed with the row at any vertex  $A$  of the triangle is given, and since three of its rays are given the fourth is known.]

**Def.** Lines divided equianharmonically are also said to be divided *Homographically*. The term homographic is applied in general to the equianharmonic division of figures of the same kind, *e.g.* lines, circles, etc., etc.

#### EXAMPLES.

1. Every tangent to a circle is cut harmonically by the sides of the escribed square.

[In the limiting position when the variable tangent coincides with a side of the square the row of points determined on it are harmonic ; therefore, etc., Art. 81, Ex. 3.]

2. To express the anharmonic ratios in which a variable tangent is divided by four fixed tangents, in terms of the chords of contact of the tangents.

[Let  $P, Q, R, S$  denote the points of contact of the sides of the escribed quadrilateral, which meet the variable tangent at  $O$  in  $A, B, C, D$  ;  $O'$  the centre of the circle. Then  $ABCD = O'.ABCD = O.PQRS$ , since  $O'A, OP$  ;  $O'B, OQ$  ... are four pairs of perpendicular lines ; therefore the required expressions are

$$QR.PS : RP.QS : PQ.RS.]$$

3. For any quadrilateral escribed to a circle at the points  $P, Q, R, S$ , each pair of diagonals and a corresponding pair of opposite connectors of the inscribed quadrilateral  $PQRS$  are concurrent. (See Art. 67, Cor. 8.)

[To prove that the sets of lines

$$\begin{aligned} QR, PS, YY', ZZ' \\ RP, QS, ZZ', XX' \\ PQ, RS, XX', YY' \end{aligned}$$

are each concurrent.

Consider each of the four tangents at the points  $P, Q, R, S$  a transversal to the quadrilateral  $XX'YY'ZZ'$ . Since consecutive tangents meet on the circle, the tangents at  $P$  and  $Q$  are cut in the same order at the points  $P, Z, Y, X'$  and  $Z, Q, X, Y'$ ; therefore  $[PZYX'] = [ZQXY'] = [QZY'X]$ . Hence  $PQ, YY', XX'$  are concurrent. Similarly  $RS, YY'$  and  $XX'$  are concurrent; therefore, etc.]

NOTE.—As the above properties are more generally true for the Conic, we consider an interesting case which arises in the parabola when the fourth tangent is at infinity (Art. 81). Let tangents  $AC$  and  $BC$  be drawn to a parabola at the points  $A$  and  $B$ , and a third tangent  $XY$  meeting  $BC$  and  $CA$  in  $X$  and  $Y$  respectively. Then the equianharmonic relations easily reduce to  $BX/CX = CY/AY$ ; or a variable tangent divides two fixed tangents in the same ratio. It also subtends a constant angle at the focus. Therefore *the foci of the three parabolas described to touch each pair of sides ( $b, c$ , etc.) of a triangle  $ABC$  at the extremities of the third side ( $BC$ ) are the vertices of Brocard's second triangle.*

4. If a circle touch four others the anharmonic ratios of the points of contact are equal to

$$\overline{23} . \overline{14} : \overline{31} . \overline{24} : \overline{12} . \overline{34}.$$

[By Art. 7.]

5. The anharmonic ratios of the points of contact of the nine-points-circle with the in- and three ex-circles of the triangle  $ABC$  are

$$\frac{\alpha^2 - b^2}{\alpha^2 - c^2} \quad \frac{b^2 - c^2}{b^2 - \alpha^2} \quad \frac{c^2 - \alpha^2}{c^2 - b^2}$$

[As in Ex. 4.]

6. If the anharmonic ratios of four points  $A, B, C, D$  on a circle (or conic) be denoted by  $\lambda, \mu, \nu$ , etc., to prove that the anharmonic ratios of the pencil  $P. ABCD$  are  $\lambda^2, \mu^2, \nu^2$ , etc., where  $P$  is the pole of the line  $AB$ .

[Let  $PC, PD$  meet the conic again in  $C', D'$ , and  $AB$  in  $E, G$ ; then  $CD', DC'$ , and  $AB$  are concurrent at  $F$ ; and since

$$C'. ABCD = D'. ABCD, [ABCD] = [ABEF] = [ABFG] = \lambda \text{ (say);}$$

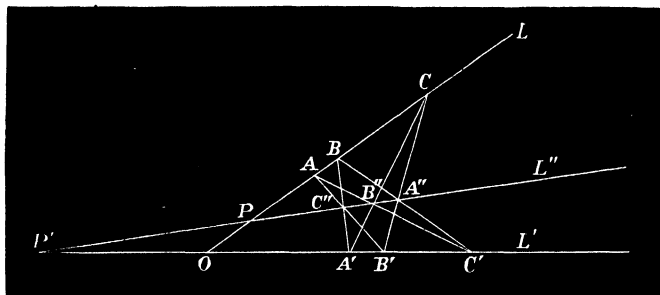
therefore

$$\frac{AE}{BE} \cdot \frac{AF}{BF} = \frac{AF}{BF} \cdot \frac{AG}{BG} = \lambda,$$

whence  $\frac{AE}{BE} \cdot \frac{AG}{BG}$  or  $[ABEG] = \lambda^2$ .

But  $[ABEG] = P \cdot ABEG = P \cdot ABCD$ ; therefore, etc.]

**137. Directive Axis.**—For any two homographic rows of points  $ABC \dots, A'B'C' \dots$  on different axes  $L$  and  $L'$ , if any pair of corresponding points  $A$  and  $A'$  be each joined to all the points on the other axis, the two pencils  $A.A'B'C' \dots, A'.ABC \dots$  are in perspective (Art. 136), i.e. the intersections of the pairs of lines  $AB', A'B(C'')$ ;  $AC'', A'C(B'')$ ;  $AD', A'D$ , etc., are collinear. We are thus enabled to find a point  $P'$  on the line  $L'$  corresponding to a given point  $P$  on  $L$ .



For having obtained the line  $B''C''$ , join  $A'P$  and let it meet  $B''C''$  in  $P''$ ; then  $AP''$  meets the axis  $L'$  in the required point.

An important point arises out of the consideration of the correspondents to the intersections  $O$ ,  $P$ , and  $P'$  of the axes  $L$ ,  $L'$ ,  $L''$  taken in pairs. By means of the general method given above we find that  $P$  on the axis  $L$  corresponds to  $O$  on the axis  $L'$ , and that  $P'$  on the axis  $L'$  corresponds to  $O$  on the axis  $L$ . This shows that the axis  $L''$  of perspective of the pencils

$$A.A'B'C' \dots, A'.ABC \dots,$$

whose vertices  $A$  and  $A'$  were arbitrarily chosen as any pair of correspondents of the given homographic systems, is a fixed line, since it meets each axis in a point corresponding to their intersection  $O$  regarded as a point on the other. Hence: all pairs of corresponding connectors ( $XY'$ ,  $X'Y$ ) of pairs of non-corresponding points lie on a line. This line is called the *Directive Axis* of the given homographic systems.

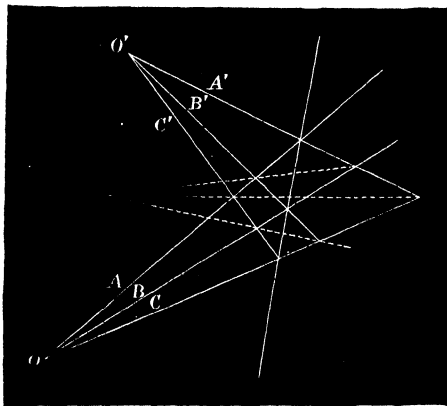
Otherwise thus: Take the two homographic pencils at  $A''$  and  $L$  and  $L'$  as transversals to them respectively, then

$$[BCPO] = [C'B'P'O];$$

similarly for the vertex  $B''$  it follows that  $[CAPO] = [A'C'P'O]$ , therefore by division (Art. 133, Ex. 3)  $[ABPO] = [B'A'P'O]$ , i.e. the lines  $AB'$ ,  $A'B$ ,  $PP'$  are concurrent.

The same proof applies to the more general case of two systems of points on a conic.

**138. Directive Centre.**—The following property of two homographic pencils is derived from Art. 137 by



reciprocation:—For any two homographic pencils of rays  $O. ABC \dots$  and  $O'. A'B'C' \dots$  the lines joining pairs of cor-

*responding intersections ( $AB'$ ,  $A'B$ ) of non-corresponding rays ( $A$ ,  $B'$  and  $A'$ ,  $B$ ) are concurrent.*

The point of concurrence is termed the *Directive Centre* of the systems, and its property just stated may be proved by methods analogous to either of those given in Art. 137 for the directive axis. These are left as useful exercises for the student.

**139. Problem.**—*To find a point  $X$  on either axis  $L$  whose correspondent on the other is at infinity ( $\infty'$ ).*

Since the lines joining  $A$ ,  $\infty'$  and  $A'$ ,  $X$  meet on the directive axis, we have the following construction:—through  $A$  draw a parallel to  $L'$ , join  $A'$  to its point of intersection with the directive axis; this line meets  $L$  in the required point.

#### EXAMPLES.

1. Having given two homographic pencils of rays at different vertices; to find a ray of either corresponding to a given one of the other.

[By means of their directive centre.]

2. If two homographic rows of points are such that the points  $\infty$ ,  $\infty'$  at infinity on the axis correspond, the lines are divided similarly.

[For  $[ABC\infty]=[A'B'C'\infty']$ , hence  $AB:BC=A'B':B'C'$ ; therefore, etc.]

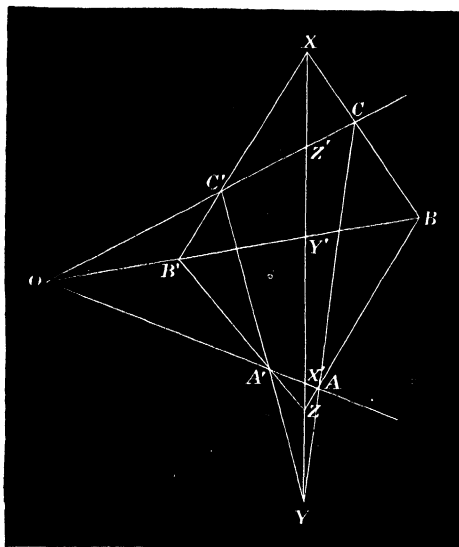
3. Having given the vertical angle in magnitude and position of a triangle of constant species, the extremities of the base divide the sides homographically.

4. If the lines  $AA'$ ,  $BB'$ ,  $CC'$  connecting the corresponding vertices of two triangles  $ABC$  and  $A'B'C'$  are concurrent at a point  $O$ , the intersections  $X$ ,  $Y$ ,  $Z$  of the pairs of sides  $BC$ ,  $B'C'$ , etc., are collinear (cf. Art. 66).

[Join  $XY$  and let it meet the lines  $AA'$ ,  $BB'$ ,  $CC'$  in  $X'$ ,  $Y'$ ,  $Z'$  respectively. Then

$$X.OBY'B = X.OCZ'C' = Y.OCZ'C' = Y.OAX'A';$$

therefore  $[OBY'B]=[OAX'A']$ , and since the point  $O$  is common to both rows the pairs of connectors  $AB, X'Y', A'B'$  are concurrent.



Therefore also the centre  $O$  and axis of perspective  $L$  of the two triangles divide the corresponding segments  $AA', BB', CC'$  equianharmonically.]

5. A variable triangle moves with its vertices on three concurrent lines such that two of its sides pass through fixed points  $X$  and  $Y$ ; then the third side passes through a fixed point on the line  $XY$ .

[By Ex. 4.]

6. The lines joining pairs of corresponding points of any two figures in perspective are cut homographically by the centre and axis of perspective.

7. Any line passing through either centre of perspective of two circles is cut in a constant anharmonic ratio by their radical axis.

8. Every four of the six points  $X, Y, Z, X', Y', Z'$  in Ex. 4 are equianharmonic with their four opposites.



9. In the figure of Art. 137 prove the relations

$$1^{\circ}. [BCPO] = [B'C'OP] = [B''C''P'P],$$

$$[CAPO] = [C'A'OP] = [C''A''P'P].$$

$$2^{\circ}. [ABCP] = [A'B'C'O] = [A''B''C''P],$$

$$[ABCO] = [A'B'C'P] = [A''B''C''P].$$

NOTE.—It will be seen that the triangle  $AB''C''$  is inscribed to  $A'BC$  and escribed to  $B'C'A''$ , and more generally that of this system of three triangles *each is inscribed to one and escribed to the other of the remaining two*.

The vertex  $A$  and opposite side  $B''C''$  of the triangle  $A'B''C''$  form with the extremities  $B$  and  $C$  of the corresponding side of  $A'BC$  to which it is inscribed a row of points  $B, C, A, P$ . Similarly the vertex  $A'$  and opposite side  $BC$  of  $A'BC$  form with the corresponding side  $B'C'$  of the triangle  $A''B''C''$  to which it is inscribed a row  $B', C', A', O$ . But these rows are equianharmonic (Ex. 8, 2°); hence for such a system of triangles *the vertex and the opposite side of each divide homographically the corresponding side of the triangle to which it is inscribed*.

Again,  $B''C''PP'$  is the row of points formed by the extremities of the base  $B''C''$  and its intersections with the corresponding sides  $BC$  and  $B'C'$  of the remaining triangles. But

$$B''C''PP' = BCPO = B'C'OP;$$

hence *the sides of each are cut homographically by the corresponding sides of the other two*.

Let the point  $C'$  vary along the axis  $L'$ . Then the lines  $AC'$  and  $BC'$  turn around the fixed points  $A$  and  $B$ ;  $A''$  and  $B''$  move on the lines  $A'C$  and  $BC''$ , and the directive axis passes through the fixed point  $C''$ . In this case  $A''B''C''$  is a variable triangle inscribed to  $A'BC$  and escribed to  $ABC''$ , both of which are fixed. Hence *for a variable triangle  $A''B''C''$  inscribed to a given one  $A'BC$ , if two of its sides pass through the vertices  $A$  and  $B$  of a triangle escribed to the latter, its third side passes through the third vertex  $C''$* .

Let us now consider two positions of the variable triangle  $A''B''C''$ . Since its sides pass through the fixed points  $A, B, C''$  respectively,  $ABC''$  is a common inscribed triangle. Hence *when two triangles are each inscribed to a third  $A'BC$ , if the sides  $A''B''$ , etc., and opposite vertices  $C''$ , etc., divide the corresponding side  $A'B'$  of  $A'BC$  in a constant*

anharmonic ratio  $[A'B'C'P]$ , the intersections of their corresponding sides determine a common inscribed triangle  $ABC''$  which is escribed to  $A'B'C'$ .

And the vertex  $C''$  and opposite side  $AB$  cut the corresponding sides  $B''C''$ , etc., in the above constant anharmonic ratio.

**140. Theorem.**—*For any two homographic rows of points  $ABC \dots X$  and  $A'B'C' \dots X'$ , if  $X$  and  $X'$  be the points whose correspondents  $\infty'$  and  $\infty$  are at infinity; to prove the relations*

$$AX \cdot A'X' = BX \cdot B'X' = CX \cdot C'X' = \text{etc.}$$

Since  $A, A'; B, B'; X, \infty'; \infty, X'$  are four pairs of corresponding points  $[ABX\infty] = [A'B'\infty'X']$ . Expanding and reducing, this relation becomes  $AX/BX = 1 \div A'X'/B'X'$ ; therefore  $AX \cdot A'X' = BX \cdot B'X'$ , etc., etc.; or:—*If variable points  $A$  and  $A'$  be taken on fixed lines  $L$  and  $L'$  respectively such that the rectangle under the distances from two fixed points  $X$  and  $X'$  on the lines is constant, they describe homographic systems.*

**COR. 1.** When the vertical angle of a triangle of constant area is given in magnitude and position, the extremities of the base divide the sides homographically.

In this case the points  $X$  and  $X'$ , whose correspondents are  $\infty'$  and  $\infty$ , are supposed to coincide at the intersection of the axes.

By Art. 81, Ex. 3, we see that the envelope of the base is a conic; and by Ex. 29 of the same article the curve is a hyperbola whose asymptotes are the given axes.

**COR. 2.** Any two homographic rows of points may be so placed that the corresponding segments  $AA', BB'$ , etc., may have a common segment of harmonic section.

Place the systems so that the axes  $L$  and  $L'$  and the

points  $X$  and  $X'$  are coincident. The equations of the article are then written

$$XA \cdot XA' = XB \cdot XB' = XC \cdot XC' = \pm \rho^2.$$

Describe a circle with  $X$  as centre having  $A, A'; B, B';$  etc., pairs of inverse points, and let it cut the axis in  $M$  and  $N$ .  $MN$  is the common segment of harmonic section by Art. 70, but it is imaginary when  $A$  and  $A'$  lie in opposite directions from  $X$ .

**Def.** Two homographic systems of points on any axis which have a common segment of harmonic section are said to be in *Involution*, and the corresponding points  $A, A'; B, B';$  etc., are *Conjugate Points* of the Involution. We have seen in Cor. 2 that there always exists a pair of points, real or imaginary, each of which regarded as belonging to either system is coincident with its correspondent of the other. These are the *Double Points* ( $M, N$ ) of the involution, and are connected with the systems by the equations

$$[MNBC] = [MNB'C'], [MNCD] = [MNC'D'], \text{ etc., etc.,} \\ [MABC \dots] = [MA'B'C' \dots] \text{ and } [NABC \dots] = [NA'B'C' \dots].$$

See Art. 133, Ex. 7.

**COR. 3.** In any two homographic rows of points on a common axis the double points  $M$  and  $N$  are found from the equations \*

$$XA \cdot XA' = XB \cdot XB' \dots = XM \cdot X'M = XN \cdot X'N;$$

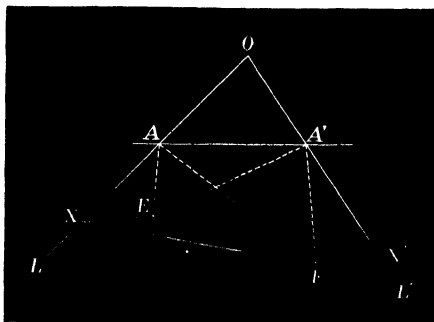
they are therefore equidistant from  $X$  and  $X'$ .

\* If the distances  $OA, OA'$  from any point  $O$  on the axis be  $x, x'$ , it follows that  $(x - OX)(x' - OX') = \text{const.}$ , a result of the form

$$Axx' + Bx + Cx' + D = 0 \text{ (cf. Art. 143).}$$

141. For any two homographic rows of points we have seen how to find the correspondent  $P'$  of any point  $P$ ,  $\alpha^\circ$ , by means of the directive axis, Art. 137, and  $\beta^\circ$  by the formula  $XP \cdot X'P' = \text{const.}$  It will now be proved that two given homographic rows can be generated by the revolution of either of two determinate angles around fixed vertices, the positions of the latter and the magnitude of the angles depending on the equal values  $[ABCD\dots]$  and  $[A'B'C'D'\dots]$  and the positions of the axes.

142. **Problem.**—*If  $ABC\dots$  and  $A'B'C'\dots$  be any two homographic rows of points; to find two points such that the angles subtended at them by the segments  $AA'$ ,  $BB'$ , etc., joining pairs of corresponding points are equal.*



Let  $E$  and  $F$  be the required points;  $X$ ,  $X'$  the correspondents of  $\infty'$  and  $\infty$  (Art. 139). Since  $AEA'$  is a constant angle, if any point  $P$  on  $L$  coincides with  $X$ ,  $EP'$  is parallel to the axis  $L'$ . Similarly if  $Q'$  and  $X'$  coincide,  $EQ$  is parallel to  $L$ . Hence the lines  $EX$  and  $BX'$  are equally inclined to  $L$  and  $L'$ , or the angles  $AXE$  and  $A'X'E$  are equal.

Again, the angles subtended at  $E$  by any two points  $A$  and  $X$  and their correspondents  $A'$  and  $\infty'$  are equal (hyp.); therefore in the two triangles  $AEX$  and  $EA'X'$  we also have the angles  $AEX$  and  $EA'X'$  equal, and the triangles are similar. Hence (Euc. VI. 4)

$$AX/XE = EX'/X'A'$$

and  $EX \cdot EX' = AX \cdot A'X' = \text{const.}$  (Art. 140).

Now in the triangle  $XXE'$  we are given the base  $XX'$  fixed, the difference of base angles and rectangle under the sides; therefore the vertex  $E$  is one or other of two fixed points  $E$  or  $F$ , which are obviously the opposite vertices of a parallelogram with  $XX'$  as diagonal.

COR. 1. The angles  $AEA'$ ,  $AXF$ , and  $A'X'F$  are equal.

For if  $A'$  and  $X'$  coincide,  $EA$  is parallel to  $L$ ; therefore  $AEA'$  is equal to the angle between  $EX'$  and  $L$  or between  $FX$  and  $L$ , since  $EX'$  and  $FX$  are parallel.

COR. 2. The triangles  $AEA'$ ,  $AXF$ , and  $EX'A'$  are similar.

[For by similar triangles  $AEX$  and  $EA'X'$  we have  $AX/AE = EX'/EA'$ , but  $EX' = FX$ , hence

$$AX/AE = FX/EA',$$

or by alternation  $AX/XF = AE/EA'$ ; therefore, etc. (Euc. VI. 6).]

COR. 3. If  $O$  denote the point of intersection of the axes  $L$  and  $L'$ , the points  $E$  and  $F$  are isogonal conjugates with respect to the variable triangle  $OAA'$ .

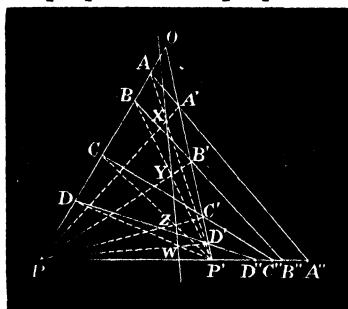
[By Cor. 2,  $FAX = EAA'$  and  $FA'X' = EA'A$ ; therefore, etc.]

COR. 4.\* The product of the perpendiculars  $p$  and  $p'$  from  $E$  and  $F$  on the variable line  $AA'$  is constant ( $pp' = k^2$ ). [By Cor. 3.]

COR. 5.\* The locus of the intersection of every two rectangular positions of  $AA'$  is a circle the square of whose radius ( $\rho$ ) is given by the equation  $\rho^2 = 2k^2 + \delta^2$ , where  $2\delta = EF$ .

COR. 6. A variable line cutting two fixed lines homographically cuts all positions of itself in a system of points  $A''B''C''\dots$  such that

$$[ABCD \dots] = [A'B'C'D' \dots] = [A''B''C''D'' \dots].$$



Draw the directive axis  $XYZ \dots$  of the system as in figure. Then  $OX$  and  $OA''$ , divide the angle  $LOL'$  of the quadrilateral  $PXP'O$  harmonically (Art. 68). Similarly for  $OY$  and  $OB'' \dots$ . Hence we have  $[O.XY \dots] = [O.A''B'' \dots]$  Art. 133, Ex. 7. But

$$[O.XY \dots] = [P.XY \dots] = [P.A'B' \dots].$$

Therefore  $[A'B'C' \dots] = [A''B''C'' \dots]$ .

\* These properties respectively may be otherwise stated :—A variable line  $AA'$  cutting two fixed axes homographically envelopes a conic of which  $E$  and  $F$  are the foci. The locus of intersection of rectangular tangents is a circle (the Director Circle).

COR. 7. If a variable line meet two fixed circles in a harmonic row of points, it intersects all positions of itself homographically.

[For the rectangle under its distances from the centres of the circles is constant, Art. 78, Ex. 12; therefore, etc., Cor. 4.]

COR. 8. A variable line meeting two fixed circles such that the chords intercepted by them are in a fixed ratio cuts all positions of itself homographically.

[By Art. 90, Ex. 8.]

143. If the distances of any point  $O$  from four points  $A, B, C, D$  on a line  $L$  passing through it be denoted by  $\alpha, \beta, \gamma, x$ , and the distances of any point  $O'$  measured along another line  $L'$  to  $A', B', C', D'$  be similarly  $\alpha', \beta', \gamma', x'$ , the two systems of points are homographic if

$$\frac{(\beta - \gamma)(\alpha - x)}{(\gamma - \alpha)(\beta - x)} = \frac{(\beta' - \gamma')(\alpha' - x')}{(\gamma' - \alpha')(\beta' - x')},$$

which when multiplied out is of the form

$$Axx' + Bx + Cx' + D = 0, \dots\dots\dots(1)$$

an equation which enables us to determine the position of any point of either system corresponding to a given one in the other. (See Art. 140, Cor. 3.)

We have seen that the lines joining corresponding points envelopes a conic touching  $L$  and  $L'$ . In the particular case when  $x = \infty$  in (1) the simultaneous value of  $x'$  is also  $\infty$ , and the corresponding conic is therefore touched by the line at infinity. It follows obviously that when  $A = 0$  in the above equation the conic is a parabola.

Thus if a variable line be drawn cutting the sides  $a$

and  $b$  of a triangle  $ABC$  in  $X$  and  $Y$  such that

$$lAY + mBX = \text{const.},$$

it envelopes a parabola to which the two sides of the triangle are tangents.

If the axes  $L$  and  $L'$  are coincident and  $B=C$  in (1),  $x$  and  $x'$  are interchangeable in the equation and, as will be more fully explained in the next chapter, the two systems are in Involution.

The double points of two systems on a common axis are found from (1) by putting  $x=x'$ , in which case the equation reduces to the form  $Ax^2 + (B+C)x + D = 0$ .

#### EXAMPLES.

1. If the distances of two pairs of collinear points  $A, B$  and  $A', B'$  from an origin  $O$  on the line be denoted by the roots of the equations  $ax^2 + 2bx + c = 0$  and  $a'x^2 + 2b'x + c' = 0$ , they form a harmonic row if  $ac' + a'c - 2bb' = 0$ .

2. Having given two of the anharmonic ratios of four collinear points equal, prove that

$$(\beta - \gamma)^2(\alpha - \delta)^2 + (\gamma - \alpha)^2(\beta - \delta)^2 + (\alpha - \beta)^2(\gamma - \delta)^2 = 0.$$



## CHAPTER XIII.

### INVOLUTION.

144. When of two systems of points  $A, B, C, \dots; A', B', C', \dots$  on any line or circle any three pairs  $A, A'; B, B'; C, C'$  which correspond are connected by a relation of the form  $[BCAA'] = [B'C'A'A]$ , it has been proved in Art. 133, Ex. 9, 1°. that every four and their four opposites are equianharmonic; 2°. that  $AA', BB', CC', \dots$  have a common segment of harmonic section.

By Art. 140, Def., we may therefore regard either of these properties as a criterion of points in Involution.

Now since  $[BCA'B'] = [B'C'AB]$ , by expanding and reducing we get

$$\frac{BA'}{CA'} \cdot \frac{CB'}{AB'} \cdot \frac{AC'}{BC'} \dots \dots \dots (1)$$

a result previously arrived at in Art. 64, where it was shown by the application of Ceva's Theorem that a straight line drawn across a quadrilateral is cut in involution; the conjugate points  $A, A'$ , etc., being the intersections of the line with the pairs of opposite connectors of the figure.

Again, if a pencil of six rays be taken and a circle described through the vertex cutting the rays in points

$A, A'; B, B'; C, C'$ , they form a system in involution if

$$\frac{\sin BOA'}{\sin COA'} \cdot \frac{\sin COB'}{\sin AOB'} \cdot \frac{\sin AOC'}{\sin BOC'} = 1 \dots \dots \dots (2)$$

The criteria (1) and (2) are called *Equations of Involution*.

145. It has been noticed in Art. 134, Ex. 10, that when any two conjugates  $A$  and  $A'$  of two homographic systems are interchangeable, every two are interchangeable, and  $AA', BB', CC' \dots$  have a common segment or angle of harmonic section.

It follows that "when any one point on an axis, or ray through a vertex, has the same correspondent to which-ever system it be regarded as belonging, then every point on the axis or ray through the vertex possesses the same property." \*

In illustration of this theorem, let the correspondents be joined in pairs to any point ( $A''$ ) on the directive axis of the systems (Art. 137).

Then the corresponding rays  $A''B, A''B'$  are interchangeable, their productions through  $A''$  being  $A''C', A''C$ ; therefore

*The locus of a point at which two homographic rows subtend a pencil in involution is their directive axis; and similarly, or by reciprocation, a variable line meeting two homographic pencils at a system of points in involution passes through their directive centre.*

146. A system of points in involution on a line is completely determined when two pairs of its conjugates  $A, A'; B, B'$  are given; and the conjugate  $C'$  of any point

---

\* Townsend, *Modern Geometry*, vol. ii. p. 276.

$C$  is its inverse with respect to the circle described with  $AB$  and  $A'B'$  as a pair of inverse segments.

If the radius of the circle is indefinitely great, one of the double points ( $N$ ) is at infinity, and therefore (Art. 72, Cor. 3)  $MA = MA'$ ,  $MB = MB'$ , etc., etc.; that is, *if one of the double points of a system in involution is at infinity, the segments  $AA'$ ,  $BB'$ ,  $CC'$  ... have a common centre, viz., the other double point.*

Also a variable segment  $AA'$  of constant length moving along a given axis determines two systems of points in involution the double points of which are imaginary.

**147. Theorem.**—*If two chords  $AA'$ ,  $BB'$  of a circle meet in  $C$ , any line through  $C$  which meets the circle in  $O$  and  $O'$  determines a system of points  $A$ ,  $A'$ ;  $B$ ,  $B'$ ;  $O$ ,  $O'$  in involution.*

Let  $AB$  and  $OO'$  meet in  $Z$  (Art. 64, iv. fig.). Then the pencil  $B. AB'OO'$  is equianharmonic with the row of points  $ZCOO'$  it determines on the transversal to it through  $C$ . For a similar reason

$$[ZCOO'] = A . BA'OO' = [A'BO'O],$$

from which relation it follows that every four of the six cyclic points and their four opposites are equianharmonic.

The concurrency of the chords  $AA'$ ,  $BB'$ ,  $OO'$ , being involved in this relation, furnishes a geometrical explanation of the theorem of Art. 133, Ex. 9 (1).

The following generalized statement is a direct inference of the preceding:—

*If through any point  $P$ , inside or outside a circle (or conic) a number of chords be drawn to cut the curve in  $A$ ,  $A'$ ;  $B$ ,  $B'$ ;  $C$ ,  $C'$ , ..., the two systems  $ABC$  ...,  $A'B'C'$  ...*

are in involution, and (Art. 64, III.) the polar of  $P$  meets the circle in the double points, real or imaginary.\*

EXAMPLES.

1. A variable line passing through either centre of similitude of two circles cuts them in four equianharmonic systems of points.

2. A variable circle cutting two given ones at equal or supplemental angles divides them equianharmonically.

3. If two circles  $V_1, V_2$  cut two others at the same angles  $\alpha$  and  $\beta$  in the points  $A, B, C, D$  and  $A', B', C', D'$ , prove that

$$[ABCD] = [A'B'C'D'].$$

$[AA', BB', CC', DD']$  are concurrent at the external centre of similitude of  $V_1, V_2$ . Cf. Art. 113, Ex. 12.]

4. More generally for any number of circles  $V_1, V_2, \dots V_n$ , prove that  $[AA'A''\dots] = [BB'B''\dots] = [CC'C''\dots] = [DD'D''\dots]$ .

5. In Ex. 3, if the angles  $\alpha$  and  $\beta$  are right, the anharmonic ratio of the four points of intersection of the variable circle is equal to that of the four points on their common diameter.

6. If two triangles  $ABC, A'B'C'$  inscribed in the same circle are in perspective at  $O$ , and from any point  $P$  on the circle lines  $PA', PB', PC'$  are drawn meeting the sides of  $ABC$  in  $X, Y, Z$ , the points  $X, Y, Z, O$  are collinear.

[The Pascal hexagons  $PB'BACC', PC'CBAA', PA'ACBB'$  have  $YOZ, ZOZ, XOY$  as Pascal lines; therefore, etc.]

7. If  $P$  denote the point on the circle corresponding to  $P$  in the perspective, and the lines  $PA, PB, PC$  meet the sides of  $A'B'C'$  in  $X', Y', Z'$ , 1°.  $X', Y', Z'$  are collinear with  $X, Y, Z$  and the six points are in involution; 2°.  $[XYZO] = [X'Y'Z'O]$ .

(Townsend, vol. ii. p. 208.)

---

\* When the point is outside its polar cuts the circle in real points  $M$  and  $N$  which divide  $AA', BB', CC'$ ... harmonically, and are therefore the double points of the involution  $ABC \dots, A'B'C' \dots$ .

† It follows directly that the anharmonic ratio of four points on a circle is unaltered by inversion; the circle of inversion in this case being either circle of antisimilitude of  $V_1$  and  $V_2$ .

8. A variable circle cutting three fixed circles at equal or similar angles determines six homographic systems of points on the circles.

[Take two positions of the variable circles cutting the given ones at equal angles  $\alpha$  and  $\beta$  respectively; then each of the given ones cuts a coaxial system (Art. 114, Ex. 10) at the same angles  $\alpha$  and  $\beta$ ; therefore, etc. It is evident that the three pairs of double points of the homographic systems on each circle are the points of contact of the corresponding circles of contact.]

9. Describe a circle touching three given ones with contacts of assigned species. [By Ex. 7.]

10. Describe a circle passing through a fixed point and cutting two given arcs on each of two circles equianharmonically.

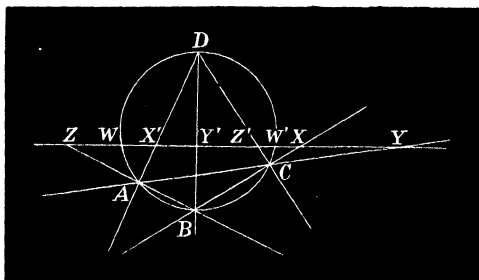
11. Describe a circle cutting three pairs of arcs on three given circles equianharmonically.

12. The line joining the centres of perspective of any two chords of a circle is divided harmonically both by the circle and the chords.

13. Equal arcs of a circle are divided equianharmonically by the two circular points at infinity.

### DESARGUES' THEOREM.

148. *Any transversal to a cyclic quadrilateral  $ABCD$  meets the three pairs of opposite connectors  $BC$  and  $AD$ ,*



*etc., etc., in  $X, X'$ ;  $Y, Y'$ ;  $Z, Z'$  and the circle in  $W$  and  $W'$  in eight points in involution.*

For the pencils  $B. ADWW'$  and  $C. ADWW'$  are equal, and therefore  $[ZY'WW'] = [YZ'WW'] = [ZYW'W]$ , or the two triads  $Y, Z, W; Y', Z', W'$  are in involution.

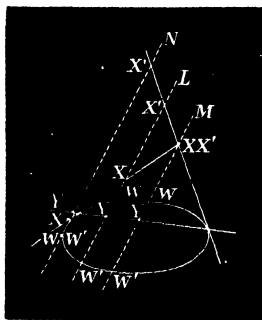
Again, because  $C. BDWW' = A. BDWW'$  it follows similarly that  $Z, X, W$  and  $Z', X', W'$  are in involution; and since  $A. CDWW' = B. CDWW'$ ,  $X, Y, W$  and  $X', Y', W'$  are in involution; therefore, etc., Art. 144.

COR. 1. By reciprocation with respect to the given circle we obtain the correlative theorem:—

*For any escribed quadrilateral the lines joining any point  $P$  to the three pairs of opposite intersections  $X, X'; Y, Y'; Z, Z'$  and the pair of tangents  $PW, PW'$  are in involution.*

COR. 2. By reciprocation from any origin it follows that the theorem and Cor. 1 are more generally true for a quadrilateral inscribed or escribed to conic.

COR. 3. In the particular case when a pair of opposite sides of a cyclic quadrilateral, or one inscribed in a conic, coincide, the remaining pair become tangents, and the

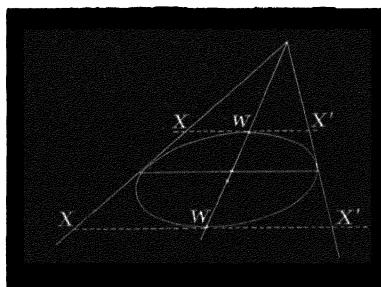


transversal ( $L$ ) meets their chord of contact in a double point.

Also the line ( $M$ ) passing through their point of intersection, which is therefore a double point, is divided harmonically; *i.e.* *A variable chord of a conic passing through a fixed point is divided harmonically by the point and its polar.*

COR. 4. When the transversal ( $N$ ) is a tangent to the conic, the points of contact ( $WW'$ ) and ( $YY'$ ) are the double points.

COR. 5. As a particular case of Cor. 4, let the transversal be parallel to the chord of contact. Then one of the double points ( $YY'$ ) is at infinity, and the other is there-



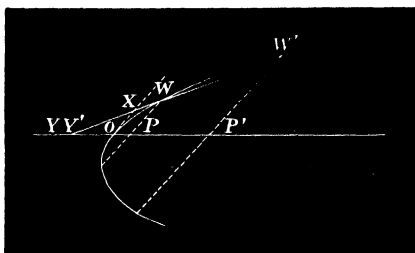
fore the middle point of  $XX'$ , hence we have the following property:—

The chord of contact of two parallel tangents (*i.e.* a diameter) bisects every parallel chord of the conic, or *the locus of the middle points of parallel chords of a conic is a right line.*

COR. 6. Since a parabola touches the line at infinity (Art. 81) and the chord of contact of any tangent and the line at infinity is a diameter, any chord ( $WW'$ ) of a parabola meets a tangent at a point  $X$ , which is the centric,

and the diameter through its point of contact at a double point ( $YY'$ ) of the involution. Hence also

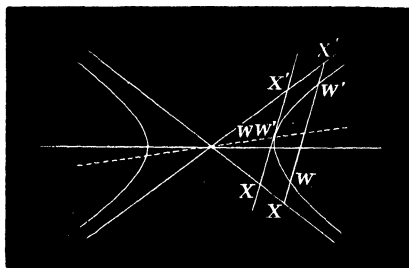
$$XW \cdot XW' = XY^2,$$



or by drawing the ordinates  $WP, W'P'$ ,

$$OP \cdot OP' = OY^2.$$

COR. 7. Since the asymptotes of a hyperbola and the line at infinity are a particular case of a quadrilateral inscribed in a conic, *any transversal  $WW'$  is divided*



*similarly at  $X$  and  $X'$ , because one of the double points ( $YY'$ ) is at infinity. The other double point is therefore the middle point of  $WW'$ , and the intercepts  $WX$  and  $W'X'$  between the curve and the asymptotes are equal.*

Also, the portion of any tangent to a hyperbola intercepted by the asymptotes is bisected at the point of contact.



COR. 8. If the point  $P$  in Cor. 1 is such that two pairs of opposite connectors  $PX, PX'$ ;  $PY, PY'$  are at right angles, the tangents from  $P$  to the circle are likewise at right angles. But the circle reciprocates from  $P$  as origin into an equilateral hyperbola; therefore *if an equilateral hyperbola be circumscribed to a triangle, it passes through the orthocentre.*

More generally, *if an equilateral hyperbola be described about a quadrilateral, it passes through the orthocentre of the four triangles formed by taking the vertices in triads.*

The property of Art. 68, Ex. 8, will now appear obvious.

It follows also that *the locus of the centres of equilateral hyperbolas described about a triangle is its nine-points-circle.*

COR. 9. If the sides of the quadrilateral be numbered 1, 2, 3, 4, and the perpendiculars from  $W$  and  $W'$  on them be denoted by  $p_1, p_2, p_3, p_4$ ;  $q_1, q_2, q_3, q_4$ , since

$$[WW'XX'] = [WW'YY'] = [WW'ZZ'],$$

and therefore  $\frac{W'X}{WX} \cdot \frac{WZ'}{W'Z} = \frac{WZ}{W'Z} \cdot \frac{WX'}{W'X}$  etc., etc.,

we have  $\frac{p_2 p_3}{q_2 q_3} = \frac{p_1 p_4}{q_1 q_4}$ ;

hence  $p_2 p_3 / p_1 p_4$  is of constant value for all points on the conic, or *the locus of a point such that the products of the perpendiculars from it to the three pairs of opposite sides of a quadrilateral have constant ratios is a conic passing through its vertices*; and by reciprocation we derive the correlative theorem:—*If a quadrilateral is circumscribed*

*of opposite vertices from a variable tangent have are to each other in constant ratios.\**

COR. 10. If either asymptote of a hyperbola be taken as a transversal to an inscribed quadrilateral, the double points of the involution are both at infinity, and the segments  $XX'$ ,  $YY'$ ,  $ZZ'$  have a common middle point; therefore *the lines joining a variable point on a hyperbola to a pair of fixed points on it intercept segments of constant length on each of the asymptotes.*

This property is thus stated in Townsend's *Modern Geometry*, Art. 340 :—

“For every two homographic pencils of rays through different vertices there exist two lines, real or imaginary, on each of which the several pairs of corresponding rays intercept equal segments.”

#### EXAMPLES.

1. A pencil whose rays are parallel to the three pairs of opposite connectors of a quadrilateral determines a system in involution.

[Since the line at infinity is a transversal cut in involution by the sides of the quadrilateral; therefore, etc.]

2. The three pairs of parallels drawn through the vertices and the extremities of the third diagonal of a quadrilateral cut any transversal in a system of points in involution.

3. If the fourth vertex  $D$  of the quadrilateral  $ABCD$  is the orthocentre of  $ABC$ , prove the following particular case of the general theorem of Art. 148 :—*For any pencil of rays in involution, if two pairs of conjugates are at right angles, then all pairs of conjugates are at right angles.*

4. Hence deduce “*The circles on the diagonals of a complete quadrilateral are coaxal.*”

•

---

\* Chasles, *Sectiones coniques*, Art. 26.

5. Any line or circle intersects a coaxal system at points in involution.

6. The parallels through any point to the sides of a triangle and the lines connecting that point to the vertices form an involution.

7. Every two circles and their two centres of perspective subtend at any point a pencil in involution.

8. For every two self-reciprocal triangles with respect to the same circle any two vertices connect equianharmonically with the remaining four.

## CHAPTER XIV.

### DOUBLE POINTS.

149. The solutions of a large number of problems of every variety in Geometry are frequently made to depend on the finding of the double points of two homographic systems. On account of the great importance of these points various constructions have been given for them. Thus in the last corollary they are easily found when we have obtained the points whose conjugates are at infinity on the axis by the equations

$$XA \cdot X'A' = XM \cdot X'M = XN \cdot X'N.$$

We give in the following article two additional constructions for homographic rows on an axis and append a sufficient number of examples, some of which have apparently no connexion with our present subject, to enable the student to form an idea of their extensive applications.

150. For any two systems of points on a circle (Art. 67, Ex. 6) the pairs of lines  $BC', B'C$ ;  $CA', C'A$ ;  $AB', A'B$  intersect respectively in points  $X, Y, Z$ , which are collinear; and the line of collinearity meets the circle in points  $M$  and  $N$ , real or imaginary, given by the equations

$$[ABCM] = [A'B'C'M] \text{ and } [ABCN] = [A'B'C'N].$$

But since the anharmonic ratios are unaltered by inversion, if the origin  $O$  be taken on the circle, the cyclic system inverts into points lying on a line and the double points of the former invert into the double points of the latter system.

Hence the following construction for the double points of two homographic systems  $ABC \dots$  and  $A'B'C' \dots$  on a line.

Take any arbitrary point  $O$  and describe the circles  $BOC'$ ,  $B'OC$  meeting again in  $X$ ;  $COA'$ ,  $C'OA$  in  $Y$ ; and  $AOB'$ ,  $A'OB$  in  $Z$ . Then  $O$ ,  $X$ ,  $Y$ ,  $Z$  lie on a circle which meets the axis in the required points  $M$  and  $N$ , real or imaginary. (Chasles.)

Otherwise thus:—Since  $[BCAM] = [B'C'A'M]$ , we have

$$\frac{BA}{CA} \bigg/ \frac{BM}{CM} = \frac{B'A'}{C'A'} \bigg/ \frac{B'M}{C'M},$$

which gives on reduction the ratios  $MB \cdot MC' / MB' \cdot MC$ , a known quantity.

But the numerator and denominator are respectively the squares of the tangents from  $M$  to the circles described on the segments  $BC'$  and  $B'C$  as diameters; therefore, etc., by Art. 88, Cor. 2.

It should be noticed that two homographic systems *whose double points are imaginary* may be generated by the revolution of a constant angle about either of two fixed vertices which are reflexions of one another with respect to the axis. For if  $AA'$ ,  $BB'$ , and  $CC'$  subtend equal angles at a point  $P$  (Art 72, Cor. 8), then

$$DPD' = APA' = \text{etc.},$$

since  $[ABCD \dots] = [A'B'C'D' \dots]$ .

EXAMPLES.

1. Through a given point  $P$  draw a line meeting two given lines  $L$  and  $L'$  divided homographically in corresponding points  $A', A''$ .

[Join  $PA, PB, PC$ , and let these lines meet the axis  $L'$  in  $A'', B'', C''$ , then  $ABC... = A''B''C''...$  since the systems are in perspective at  $P$ , therefore  $A''B''C''... = A'B'C'...$ , and if any point of either coincides with its correspondents of the other, what is required is done; hence lines joining  $P$  to the double points of these systems give the two solutions of the problem.]

2. Draw a line through a point  $P$  cutting four lines  $L_1, L_2, L_3, L_4$  in a row of points  $A, B, C, D$  having a given anharmonic ratio  $k$ .

[Take points  $A_1, A_2, A_3, \dots$  on the axis  $L_1$ , and draw lines cutting the remaining axes in systems of points such that

$$A_1B_1C_1D_1 \dots = A_2B_2C_2D_2 \dots = A_3B_3C_3D_3 \dots$$

The angle  $L_1L_2$  is thus divided homographically by the pairs of rays through  $C_1, D_1; C_2, D_2; C_3, D_3 \dots$ , etc., and the systems  $C_1C_2C_3 \dots, D_1D_2D_3 \dots$  are therefore equianharmonic.\* Join  $PC_1, PC_2, PC_3, \dots$ , and let the joining lines meet  $L_4$  in  $D'_1, D'_2, D'_3, \dots$ . It follows, as in Ex. 1, that  $D_1D_2D_3 \dots = D'_1D'_2D'_3 \dots$ , and the lines joining their double points to  $P$  are those required.]

3. Draw a line intersecting five lines such that the anharmonic ratio of any four of the points of intersection is equal to that of any other four.

4. Given two homographic pencils, find the pairs of corresponding rays which intersect on a given line  $L$ .

[Let the line meet the pencils in points  $ABC, A'B'C'$ ; the required rays therefore pass through the double points of the homographic rows so determined.]

5. In Ex. 4 find the pair of corresponding rays which intersect at a given angle.

[Join the vertices  $O$  and  $O'$  of the pencils, and on  $OO'$  describe a segment of a circle containing the given angle; let this circle cut the pencils in the points  $ABC \dots, A'B'C' \dots$ , and find the double points of these homographic systems; therefore, etc.]

---

\* This is otherwise evident as all the lines touch the same conic.

6. Find the direction of the parallel rays; and hence draw a transversal to two homographic pencils which shall be divided similarly by them.

7. Find two points on a given line which shall be isogonal conjugates with respect to a given triangle.

8. Construct a triangle with its sides passing through given points and its vertices on given lines, or on a circle.

9. Let the line  $L$  joining the vertices of two homographic pencils regarded as a ray of each system have for conjugates  $L_1$  and  $L_2$ ; prove that any transversal through the point  $L_1L_2$  is cut in involution (cf. Art. 145).

10. Through a given point  $P$  draw a line intersecting five lines in the points  $A, A'; B, B'; P'$  in any assigned order forming with  $P$  an involution.

[Let the lines containing  $A, B$  meet in  $O$ ; those containing  $A', B'$  in  $O'$ . Since (hyp)  $OABPP' - O'A'B'P'P - O'B'A'PP'$  and the pairs of rays which correspond  $OB, O'B'$ ;  $OB, O'A'$  are fixed; therefore the variable rays  $OP'$  and  $O'P'$  divide the fifth line  $L$  homographically and the double points give the required solutions.]

11. Find a point on a given line such that if joined to five given points any two pairs of connectors shall be in involution with the line and fifth.

12. Describe a circle touching three circles with contacts of assigned species.

# INDEX.

	PAGE		PAGE
ANGLE, Brocard (of a Tri- angle), - - -	61, 105	CIRCLE of Apollonius, - -	148
Brocard (of a Poly- gon), - - -	252	Auxiliary, - - -	170
of Intersection of Figures, - - -	6	Brocard's, - - -	131
Antisimilitude, Products of,	222	Coaxal, - - -	168
Antiparallels, - - -	68	Concentric, - - -	167
AXIS, Central, - - -	107	Cosine, - - -	74
Directive, - - -	270	Director, - - -	171, 279
of Perspective, - - -	121	M'Cay's ("A," "B," "C," circles),	211
Radical, - - -	143	Neuberg's, - - -	131
Barbarin, - - -	81	Nine Points, - - -	70, 86
Biddle, Mr. D., - - -	44	Orthocentroidal, - -	200
Bowesman, Mr. George W.,	203	Polar, - - -	149
Burnside, Professor, - -	33	of Self-Inversion, -	246
Brianchon, - - -	167	of Similitude, - -	197
Brocard, M., - - -	93	Taylor's, - - -	76
Casey, - - -	20, 217, 218, 241	Triplicate Ratio ("T.R." circle),	75
Catalan, - - -	4	Tucker's, - - -	71
Chasles, - - -	50	Conjugate Segments, -	261
CENTRE Directive, - -	271	Desargues, - - -	115, 286
Instantaneous, - - -	206	Diameter (of a Polygon),	108
Perspective, - - -	120	Dilworth, Mr. William J. (T. C. D.), - - -	144
Radical, - - -	143	Envelopes, - - -	9
of Similitude, 42, 52, 185		Equianharmonicisim, -	4
Chords, Homologous, and		Fry, Mr. Matthew W. J. (F.T.C.D.), - - -	260
Antihomologous, - - -	221		



	PAGE		PAGE
Hain, - - - - -	75	POINTS, Brianchon, - - -	126
Hart, Sir Andrew S., - -	187	Brocard, - - - 60,	104
HEXAGONS, Brianchon, -	126	two Circular, - - -	174
Equianharmonic, - - -	167	Conjugate, - - -	149
Pascal, - - - - -	128	Director, - - - - -	210
Homology, Axis of, - - -	121	Double, - - - - - 206,	276
Homothetic Figures, - -	204	de Gergonne, - - -	117
Homographic Division, -	268	Homologous, - - -	220
Ingram, Dr. J. K. (SF.T.C.D.),	260	Invariable, - - -	208
Involution, - - - - -	118, 276	Limiting, - - - - -	176
Equations of, - - - - -	119, 273	Middle of a Line, - -	153
Isogonal Conjugates, - -	132	de Nagel, - - - - -	117
Isotomic „ - - - - -	118	Pascal, - - - - -	124
LINES, Antihomologous, -	221	Symmedian, - - - -	31
Brianchon, - - - - -	126	Tarry's - - - - -	136
Conjugate, - - - - -	146	Polar Equations, - - -	65
Diagonal, - - - - -	190	POLYGONS, Convex, - -	133
Homologous, - - - - -	221	Intersecting, - - -	133
Pascal, - - - - -	124	Re-entrant, - - -	133
Philo's, - - - - -	37	Harmonic, - - - - -	252
Simson, - - - - -	46, 50	QUADRILATERALS, Harmonic,	
„ (pole of), - - - - -	81, 256	properties	
Longchamps, Professor de, -	182	of, - - - - -	132
Mathesis, 47, 75, 80, 81, 94,	161,	Diagonal,	
209, 260		Line of, - - - - -	192
Milne, Rev. J. J., - - -	39, 92, 252	Harmonic, - - - - -	252
Mukhopadhyay, Syamadas, -	94	Radius Vector, - - - -	43, 65
M'Cay, Mr. W. S. (F.T.C.D.),		Ratios, Anharmonic, 3, 117,	261
48, 66, 97, 183, 191, 211, 254,	256	of Similitude, - - -	42
Neuberg, M., - - - - -	64, 75, 76, 216	Reciprocation, - - - -	162
Nouvelles Annales, - - -	76	Russell, Mr. Robert	
Parallel Lines, - - - - -	9	(F.T.C.D.), - - - 190, 201,	255
Philo's Line, - - - - -	37	Salmon, Rev. Dr., - - -	263
Projection of a Point, - -	4	Simmons, Rev. T. C., - -	44
Pascal, - - - - -	167	Stubbs, Rev. Dr., - - -	260
Poncelet, - - - - -	115, 121	Steiner, - - - - -	49
Preston, Mr Thomas, - - -	196	Symmetry, - - - - -	1
POINTS, Adjoint, - - - -	209	Systems of Circles (Coaxal),	173
Antihomologous, - - - -	221	Conjugate Coaxal, - -	173

	PAGE		PAGE
Tangential Equations, -	159	THEOREMS by Tarry, -	209
THEOREMS by Bobillier, -	10	Weill, -	96, 201
Brianchon, -	125	Townsend, 151, 224, 244, 247,	283, 285
Casey, -	11	TRIANGLES, Brocard's First, 122	
Ceva, -	113	,, Second,	210, 269
Desargues, -	286	Cosymmedian, -	252
Euler, -	4	Diagonal, -	132
Feuerbach, 13, 217		Invariable, -	208
Hart, -	13	Median, -	160
Hermite, -	134	Pedal, -	22
Mannheim, 11, 255		of Reference, -	63
,, (ex-		Self-Reciprocal, 151	
tension of), -	259	Similitude, -	207
Menelaus, -	112	Tucker, - - -	83
Neuberg, -	210	Van de Berg, - - -	161
Pascal, -	124	Vigarié, - - -	253
Poncelet, -	187	Walker, - - -	189
Ptolemy, -	48	Ward, Mr. Piers C. (T.C.D.),	242
Russell, -	201	Weill, - - -	201, 242
Salmon, -	155		
Stoll, -	123		



Messrs. Macmillan and Co.'s Books for Students.

## ANALYTICAL GEOMETRY.

*EXERCISES IN ANALYTICAL GEOMETRY.* By J. M. Dyer, M.A., Assistant Master at Eton. Illustrated. Cr. 8vo. 4s. 6d.

*AN ELEMENTARY TREATISE ON TRILINEAR CO-ORDINATES*, the Method of Reciprocal Polars, and the Theory of Projectors. By the Rev. N. M. Ferrers, D.D., F.R.S., Master of Gonville and Caius College, Cambridge. 4th Ed., revised. Cr. 8vo. 6s. 6d.

*CURVE TRACING IN CARTESIAN CO-ORDINATES.* By W. Woolsey Johnson, Professor of Mathematics at the U.S. Naval Academy, Annapolis, Maryland. Cr. 8vo. 4s. 6d.

*AN ELEMENTARY TREATISE ON CONIC SECTIONS AND ALGEBRAIC GEOMETRY.* With Numerous Examples and Hints for their Solution. By G. H. Puckle, M.A. 5th Ed., revised and enlarged. Cr. 8vo. 7s. 6d.

WORKS BY PERCIVAL FROST, D.Sc., F.R.S.,  
Fellow and Mathematical Lecturer at King's College, Cambridge.

*AN ELEMENTARY TREATISE ON CURVE TRACING.* 8vo. 12s.

*SOLID GEOMETRY.* 3d Ed. Demy 8vo. 16s.

*HINTS FOR THE SOLUTION OF PROBLEMS* in the Third Edition of *SOLID GEOMETRY.* 8vo. 8s. 6d.

WORKS BY CHARLES SMITH, M.A.,  
Master of Sidney Sussex College, Cambridge.

*CONIC SECTIONS.* 7th Ed. Cr. 8vo. 7s. 6d.

*SOLUTIONS TO CONIC SECTIONS.* Cr. 8vo. 10s. 6d.

*AN ELEMENTARY TREATISE ON SOLID GEOMETRY.* 2nd Ed. Cr. 8vo. 9s. 6d.

WORKS BY ISAAC TODHUNTER, F.R.S.,  
*PLANE CO-ORDINATE GEOMETRY*, as applied to the Straight Line and the Conic Sections. Cr. 8vo. 7s. 6d.  
*KEY.* By C. W. Bourne, M.A., Headmaster of King's College School. Cr. 8vo. 10s. 6d.

*EXAMPLES OF ANALYTICAL GEOMETRY OF THREE DIMENSIONS.* New Ed., revised. Cr. 8vo. 4s.

MACMILLAN & CO., LONDON.

## EUCLID AND PURE GEOMETRY.

- A TREATISE ON GEOMETRICAL CONICS.* In accordance with the Syllabus of the Association for the Improvement of Geometrical Teaching. By A. Cockshott, M.A., Assistant Master at Eton, and Rev. F. B. Walters, M.A., Principal of King William's College, Isle of Man. Cr. 8vo. 5s.
- GEOMETRICAL EXERCISES FOR BEGINNERS.* By Samuel Constable. Cr. 8vo. 3s. 6d.
- EUCLIDIAN GEOMETRY.* By Francis Cuthbertson, M.A., LL.D. Ex. fcap. 8vo. 4s. 6d.
- PROPERTIES OF CONIC SECTIONS PROVED GEOMETRICALLY.* By Rev. H. G. Day, M.A. Part I. The Ellipse, with an ample collection of Problems. Cr. 8vo. 3s. 6d.
- RIDER PAPERS ON EUCLID. BOOKS I. AND II.* By Rupert Deakin, M.A. 18mo. 1s.
- GEOMETRICAL TREATISE ON CONIC SECTIONS.* By W. H. Drew, M.A. New Ed., enlarged. Cr. 8vo. 5s.
- ELEMENTARY SYNTHETIC GEOMETRY OF THE POINT, LINE AND CIRCLE IN THE PLANE.* By N. F. Dupuis, M.A., Professor of Pure Mathematics in the University of Queen's College, Kingston, Canada. Gl. 8vo. 4s. 6d.
- A TEXT-BOOK OF EUCLID'S ELEMENTS.* Including Alternative Proofs, together with additional Theorems and Exercises, classified and arranged. By H. S. Hall, M.A., and F. H. Stephens, M.A., Masters of the Military and Engineering Side, Clifton College. Gl. 8vo. Book I., 1s.; Books I. and II., 1s. 6d.; Books I.-IV., 3s.; Books III.-IV., 2s.; Books III.-VI., 3s.; Books V.-VI. and XI., 2s. 6d.; Books I.-VI. and XI., 4s. 6d.; Book XI., 1s. [KEY. *In preparation.*]
- THE ELEMENTS OF GEOMETRY.* By G. B. Halsted, Professor of Pure and Applied Mathematics in the University of Texas. 8vo. 12s. 6d.
- THE ELEMENTS OF SOLID GEOMETRY.* By R. B. Hayward, M.A., F.R.S. Gl. 8vo. 3s.
- EUCLID FOR BEGINNERS.* Being an Introduction to existing Text-Books. By Rev. J. B. Lock, M.A. [In the Press]
- GEOMETRICAL CONICS. Part I. The Parabola.* By Rev. J. J. Milne, M.A., and R. F. Davis, M.A. Cr. 8vo. 2s.
- THE PROGRESSIVE EUCLID. BOOKS I. and II.* With Notes, Exercises, and Deductions. Edited by A. T. Richardson, M.A., Senior Mathematical Master at the Isle of Wight College. Gl. 8vo. 2s. 6d.
- SYLLABUS OF PLANE GEOMETRY* (corresponding to Euclid, Books I.-VI.)—Prepared by the Association for the Improvement of Geometrical Teaching. Cr. 8vo. Sewed, 1s.
- SYLLABUS OF MODERN PLANE GEOMETRY.* Prepared by the Association for the Improvement of Geometrical Teaching. Cr. 8vo. Sewed, 1s.
- THE ELEMENTS OF EUCLID.* By I. Todhunter, F.R.S. 18mo. 3s. 6d. Books I. and II. 1s. KEY. Cr. 8vo. 6s. 6d.

WORKS BY VEN. ARCHDEACON WILSON, M.A.,

Formerly Headmaster of Clifton College.

- ELEMENTARY GEOMETRY.* Books I.-V. Containing the Subjects of Euclid's first Six Books. Following the Syllabus of the Geometrical Association. Ex. fcap. 8vo. 4s. 6d.
- SOLID GEOMETRY AND CONIC SECTIONS.* With Appendices on Transversals and Harmonic Division. Ex. fcap. 8vo. 3s. 6d.

MACMILLAN & CO., LONDON.

## HIGHER PURE MATHEMATICS.

- ELEMENTARY TREATISE ON PARTIAL DIFFERENTIAL EQUATIONS.* With Diagrams. By Sir G. B. Airy, K.C.B., formerly Astronomer-Royal. 2d Ed. Cr. 8vo. 5s. 6d.
- ON THE ALGEBRAICAL AND NUMERICAL THEORY OF ERRORS OF OBSERVATIONS AND THE COMBINATION OF OBSERVATIONS.* By the same. 2d Ed., revised. Cr. 8vo. 6s. 6d.
- THE CALCULUS OF FINITE DIFFERENCES.* By G. Boole. 3d Ed., revised by J. F. Moulton, Q.C. Cr. 8vo. 10s. 6d.
- THE DIFFERENTIAL CALCULUS.* By Joseph Edwards, M.A. With Applications and numerous Examples. Cr. 8vo. 10s. 6d.
- AN ELEMENTARY TREATISE ON SPHERICAL HARMONICS, AND SUBJECTS CONNECTED WITH THEM.* By Rev. N. M. Ferrers, D.D., F.R.S., Master of Gonville and Caius College, Cambridge. Cr. 8vo. 7s. 6d.
- A TREATISE ON DIFFERENTIAL EQUATIONS.* By Andrew Russell Forsyth, F.R.S., Fellow and Assistant Tutor of Trinity College, Cambridge. 2d Ed. 8vo. 14s.
- AN ELEMENTARY TREATISE ON CURVE TRACING.* By Percival Frost, M.A., D.Sc. 8vo. 12s.
- GEOMETRY OF POSITION.* By R. H. Graham. Cr. 8vo. 7s. 6d.
- DIFFERENTIAL AND INTEGRAL CALCULUS.* By A. G. Greenhill, Professor of Mathematics to the Senior Class of Artillery Officers, Woolwich. New Ed. Cr. 8vo. 10s. 6d.
- APPLICATIONS OF ELLIPTIC FUNCTIONS.* By the same. [In the Press.]
- INTRODUCTION TO QUATERNIONS,* with numerous examples. By P. Kelland and P. G. Tait, Professors in the Department of Mathematics in the University of Edinburgh. 2d Ed. Cr. 8vo. 7s. 6d.
- HOW TO DRAW A STRAIGHT LINE: a Lecture on Linkages.* By A. B. Kempe. Illustrated. Cr. 8vo. 1s. 6d.
- DIFFERENTIAL CALCULUS FOR BEGINNERS.* By Alexander Knox. Fcap. 8vo. 3s. 6d.
- THE THEORY OF DETERMINANTS IN THE HISTORICAL ORDER OF ITS DEVELOPMENT.* Part I. Determinants in General. Leibnitz (1693) to Cayley (1841). By Thomas Muir, Mathematical Master in the High School of Glasgow. 8vo. 10s. 6d.
- AN ELEMENTARY TREATISE ON THE DIFFERENTIAL CALCULUS.* Founded on the Method of Rates or Fluxions. By J. M. Rice, Professor of Mathematics in the United States Navy, and W. W. Johnson, Professor of Mathematics in the United States Naval Academy. 3d Ed., revised and corrected. 8vo. 18s. Abridged Ed. 9s.

## WORKS BY ISAAC TODHUNTER, F.R.S.

- AN ELEMENTARY TREATISE ON THE THEORY OF EQUATIONS.* Cr. 8vo. 7s. 6d.
- A TREATISE ON THE DIFFERENTIAL CALCULUS.* Cr. 8vo. 10s. 6d. KEY. Cr. 8vo. 10s. 6d.
- A TREATISE ON THE INTEGRAL CALCULUS AND ITS APPLICATIONS.* Cr. 8vo. 7s. 6d. KEY. Cr. 8vo. 10s. 6d.
- AN ELEMENTARY TREATISE ON LAPLACE'S, LAME'S AND BESSEL'S FUNCTIONS.* Cr. 8vo. 10s. 6d.

MACMILLAN & CO., LONDON.

# TRIGONOMETRY.

- AN ELEMENTARY TREATISE ON PLANE TRIGONOMETRY.* With Examples. By R. D. Beasley, M.A. 9th Ed., revised and enlarged. Cr. 8vo. 8s. 6d.
- FOUR-FIGURE MATHEMATICAL TABLES.* Comprising Logarithmic and Trigonometrical Tables, and Tables of Squares, Square Roots, and Reciprocals. By J. T. Bottomley, M.A., Lecturer in Natural Philosophy in the University of Glasgow. 8vo. 2s. 6d.
- THE ALGEBRA OF CO-PLANAR VECTORS AND TRIGONOMETRY.* By R. B. Hayward, M.A., F.R.S., Assistant Master at Harrow. [In preparation.]
- A TREATISE ON TRIGONOMETRY.* By W. E. Johnson, M.A., late Scholar and Assistant Mathematical Lecturer at King's College, Cambridge. Cr. 8vo. 8s. 6d.
- ELEMENTS OF TRIGONOMETRY.* By Rawdon Levett and A. F. Davison, Assistant Masters at King Edward's School, Birmingham. Cr. 8vo. [In the Press.]
- A TREATISE ON SPHERICAL TRIGONOMETRY.* With applications to Spherical Geometry and numerous Examples. By W. J. McClelland, M.A., Principal of the Incorporated Society's School, Santry, Dublin, and T. Preston, M.A. Cr. 8vo. 8s. 6d., or; Part I. To the End of Solution of Triangles, 4s. 6d. Part II., 5s.
- MANUAL OF LOGARITHMS.* By G. F. Matthews, B.A. 8vo. 5s. nett
- TEXT-BOOK OF PRACTICAL LOGARITHMS AND TRIGONOMETRY.* By J. H. Palmer, Headmaster, R.N., H.M.S. Cambridge, Devonport. Gl. 8vo. 4s. 6d.
- EXAMPLES FOR PRACTICE IN THE USE OF SEVEN-FIGURE LOGARITHMS.* By Joseph Wolstenholme, D.Sc., late Professor of Mathematics in the Royal Indian Engineering College, Cooper's Hill. 8vo. 5s.
- THE ELEMENTS OF PLANE AND SPHERICAL TRIGONOMETRY.* By J. C. Snowball. 14th Ed. Cr. 8vo. 7s. 6d.

## WORKS BY REV. J. B. LOCK, M.A.,

Senior Fellow and Bursar of Gonville and Caius College, Cambridge.

- THE TRIGONOMETRY OF ONE ANGLE.* Gl. 8vo. 2s. 6d.
- TRIGONOMETRY FOR BEGINNERS,* as far as the Solution of Triangles. 3d Ed. Gl. 8vo. 2s. 6d. KEY. Cr. 8vo. 6s. 6d.
- ELEMENTARY TRIGONOMETRY.* 6th Ed. (in this edition the chapter on logarithms has been carefully revised). Gl. 8vo. 4s. 6d. KEY. Cr. 8vo. 8s. 6d.
- HIGHER TRIGONOMETRY.* 5th Ed. Gl. 8vo. 4s. 6d. Both Parts complete in One Volume. Gl. 8vo. 7s. 6d.

## WORKS BY ISAAC TODHUNTER, F.R.S.

- TRIGONOMETRY FOR BEGINNERS.* 18mo. 2s. 6d. KEY. Cr. 8vo. 8s. 6d.
- PLANE TRIGONOMETRY.* Cr. 8vo. 5s. A New Edition revised by R. W. Hogg, M.A. Cr. 8vo. 5s. KEY. Cr. 8vo. 10s. 6d.
- A TREATISE ON SPHERICAL TRIGONOMETRY.* Cr. 8vo. 4s. 6d.

MACMILLAN & CO., LONDON.













